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No. 1115

INVESTIGATION OF THE STABILITY OF THE LAMINAR BOUNDARY LAYER IN A COMPRESSIBLE FLUID

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SUMMARY

In the present report the stability of two-dimensional laminar flows of a gas is investigated by the method of small perturbations. The chief emphasis is placed on the case of the laminar boundary layer.

Part I of the present report deals with the general mathematical theory. The general equations governing one normal mode of the small velocity and temperature disturbances are derived and studied in great detail. It is found that for Reynolds numbers of the order of those encountered in most aerodynamic problems, the temperature disturbances have only a negligible effect on those particular velocity solutions which depend primarily on the viscosity coefficient ("viscous solutions"). Indeed, the latter are actually of the same form in the compressible fluid as in the incompressible fluid, at least to the first approximation. Because of this fact, the mathematical analysis is greatly simplified. The final equation determining the characteristic values of the stability problem depends on the "inviscid solutions" and the function of Tietjens in a manner very similar to the case of the incompressible fluid. The second viscosity coefficient and the coefficient of heat conductivity do not enter the problem; only the ordinary coefficient of viscosity near the solid surface is involved.

Part II deals with the limiting case of infinite Reynolds numbers. The study of energy relations is very much emphasized. It is shown that the disturbance will gain energy from the main flow if the gradient of the product of mean density and mean vorticity near the solid surface has a sign opposite to that near the outer edge of the boundary layer.

A general stability criterion has been obtained in terms of the gradient of the product of density and vorticity, analogous to the Rayleigh-Tollmien criterion for the case of an incompressible fluid. If this gradient vanishes for some value of the velocity ratio of the main flow exceeding 1-1/M (where M is the free stream Mach number),

then neutral and self-excited "subsonic" disturbances exist in the inviscid fluid. (The subsonic disturbances die out rapidly with distance from the solid surface.) The conditions for the existence of other types of disturbance have not yet been established to this extent of exactness. A formula has been worked out to give the amplitude ratio of incoming and reflected sound waves.

It is found in the present investigation that when the solid boundary is heated, the boundary layer flow is destabilized through the change in the distribution of the product of density and vorticity, but stabilized through the increase of kinematic viscosity near the solid boundary. When the solid boundary is cooled, the situation is just the reverse. The actual extent to which these two effects counteract each other can only be settled by actual computation or some approximate estimates of the minimum critical Reynolds number. This question will be investigated in a subsequent report.

Part III deals with the stability of laminar flows in a perfect gas with the effect of viscosity included. The method for the numerical computation of the stability limit is outlined; detailed numerical calculations will be carried out in a subsequent report.

INTRODUCTION

In a recent paper (reference 1), one of the present authors has clarified the theory of the stability of two-dimensional parallel flows in a homogeneous viscous incompressible fluid. The experimental investigations of H. L. Dryden, G. B. Schubauer, H. K. Skramstad (reference 2) and H. W. Liepmann (reference 3) agree with the calculations made by Tollmien (reference 4), Schlichting (reference 5) and those given in the paper quoted (reference 1). Because of the increasing importance of phenomena of gas flow at high speeds, it seems natural that the investigation should be extended to cover the case of a gas, taking into account the effects of compressibility and heat transfer.

The interest in this problem is further enhanced by the fact that disturbances of finite amplitude in high-speed flows are known to have the tendency of building themselves up into shock waves. It is therefore possible that instability of high-speed laminar flows will lead to shock waves instead of turbulence.[1] Although an instability theory involving only small disturbances would not be able to settle this point, it at least paves the way to such an investigation.

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[1]This possibility was first pointed out to the authors by Doctor H. W. Liepmann.

The present report is concerned with the stability of two-dimensional laminar flows of a perfect gas, subject to small disturbances. The chief emphasis is placed on the case of the boundary layer. It is hoped that the results may throw light on the general features of the relation of compression waves with the boundary layer, that it may be known how the stability of a boundary layer is affected by the free stream velocity, and by the thermal conditions at the solid boundary.

As in the incompressible case, only small two-dimensional wavy disturbances will be considered. Unfortunately, there is no rigorous proof in the present case that these disturbances are more unstable than the three-dimensional ones. But the results of the incompressible case together with some physical considerations seem to justify such a treatment, which naturally simplifies the mathematical analysis to a large extent.

One essential difference of the present problem from the instability problem in an incompressible fluid is the presence of an appreciable interchange of mechanical and heat energies. Another is the fact that the flow velocity is of the same order of magnitude as the velocity of sound. The present investigations, however, reveal that the chief physical mechanism is not changed. That is, the stability of two-dimensional parallel flows depends primarily on the distribution of angular momentum of an element of the fluid, and on the effect of viscous forces, but not directly on heat conductivity. The viscous forces influence the stability of the flow both in building up the disturbance by increasing the Reynolds shear stress and in destroying the disturbance by dissipation. (Cf. sec. 14 of reference 1.) In the present case, however, the angular momentum of a given volume of the fluid depends upon the product of density and vorticity. Thus, the gradient of this product plays the same role as the curvature of the velocity profile (gradient of vorticity) in the incompressible case. Moreover, since the magnitude of viscosity varies with temperature across the flow, there is an uncertainty in defining a Reynolds number which will properly describe the stability characteristics under various conditions. It is concluded from the present investigations that the viscosity coefficient in the neighborhood of the solid boundary is important. This tends to justify the process of Allen and Nitzberg (reference 6) in estimating the critical Reynolds number for the boundary layer of a compressible fluid, so far as their treatment of the viscosity coefficient is concerned. They have, however, neglected the effect of the distribution of angular momentum in the fluid.

It is found in the present investigation that when the solid boundary is heated, the boundary layer flow is destabilized through the change of distribution of angular momentum, but stabilized through the increase of kinematic viscosity near the solid boundary. When the solid boundary is cooled, the situation is just the reverse. The actual extent to which these two effects counteract each other can only be settled by actual computation or some approximate estimates of the minimum critical Reynolds number. This question will be investigated in a subsequent report.

The transfer of energy between the mean flow and the disturbance is also somewhat different from that in the incompressible case, because the velocity of sound is finite. When a disturbance is being amplified, energy passes from the main flow to the disturbance not only to supply the increase of energy of disturbance inside the boundary layer, but also to supply the energy carried out of the boundary layer by the disturbance. For a damped disturbance, the opposite is true. For supersonic flows, there is also the possibility of energy transfer for neutral disturbances. The energy is carried into the boundary layer by an incoming wave and out of it by an outgoing wave. These two waves are generally present simultaneously, and the situation may be described as a reflection with a change of amplitude.

From the behavior of the disturbances outside the boundary layer, they are classified as "subsonic," "sonic," and "supersonic" disturbances, according to whether the phase velocity of the wavy disturbance in the direction of the free stream and relative to an observer moving with the velocity of the free stream, is below, equal to, or above the local velocity of sound.

The method of analysis used in the present report is very similar to that used in the incompressible case. Indeed, an attempt is made to establish results analogous to those obtained in that case. Small disturbances are considered, which are analyzed linearly into normal modes, periodic in the direction of the free stream, and these are then treated separately. The normal modes may be damped, neutral, or self-excited oscillations in time. For a given condition, the main flow is unstable if any one of these modes is self-excited. When the disturbance be-becomes so large that it can no longer be regarded as linear, the present theory ceases to apply. But it may then be expected that turbulence or shock wave will be precipitated by the nonlinear effect.

Reference should be made to the work of Schlichting (reference 7) and Küchemann (reference 8). Schlichting was interested in the stabilizing or destabilizing effect of gravity and temperature gradient. But he neglected the interchange of mechanical and heat energies. In aerodynamical problems, the authors are not particularly interested in the effect of gravity. However, in the general mathematical investigation, the inadequacy in Schlichting's fundamental equation will be discussed (sec. 2). Küchoman made only an "inviscid" investigation of the stability of the boundary layer (pt. II of this report), but he neglected the gradient of temperature and the curvature of the velocity profile. These are serious omissions. Their significance will turn out in the process of the present investigation.

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LIST OF SYMBOLS

	Dimensional quantities	Dimensionless quantities	Characteristic measure
Positional coordinates			
(1)	x^*	x	l
(2)	y^*	y	l
Time			
(3)	t^*	t	$l/\bar{u}_o^*$
Velocity components in the directions of the x- and y-axes, respectively			
(4)	$u^* = \bar{u}^* + u^{*\prime}$	$u = w(y) + f(y)e^{i\alpha(x - ct)}$	$\bar{u}_o^*$
(5)	$v^* = \bar{v}^* + v^{*\prime}$	$v = \alpha\varphi(y)e^{i\alpha(x - ct)}$	$\bar{u}_o^*$
Components of strain tensor			
(6)	$\epsilon_{xx}^* = \bar{\epsilon}_{xx}^* + \epsilon_{xx}^{*\prime}$	$\epsilon_{xx} = \bar{\epsilon}_{xx} + \epsilon_{xx}^{\prime}$	$\bar{u}_o^*/l$
(7)	$\epsilon_{xy}^* = \bar{\epsilon}_{xy}^* + \epsilon_{xy}^{*\prime}$	$\epsilon_{xy} = \bar{\epsilon}_{xy} + \epsilon_{xy}^{\prime}$	$\bar{u}_o^*/l$
(8)	$\epsilon_{yy}^* = \bar{\epsilon}_{yy}^* + \epsilon_{yy}^{*\prime}$	$\epsilon_{yy} = \bar{\epsilon}_{yy} + \epsilon_{yy}^{\prime}$	$\bar{u}_o^*/l$

	Dimensional quantities	Dimensionless quantities	Characteristic measure
Components of stress tensor			
(9)	$\tau^*_{xx} = \bar{\tau}^*_{xx} + \tau^{*\prime}_{xx}$	$\tau_{xx} = \bar{\tau}_{xx} + \tau'_{xx}$	$\bar{p}^*_o$
(10)	$\tau^*_{xy} = \bar{\tau}^*_{xy} + \tau^{*\prime}_{xy}$	$\tau_{xy} = \bar{\tau}_{xy} + \tau'_{xy}$	$\bar{p}^*_o$
(11)	$\tau^*_{yy} = \bar{\tau}^*_{yy} + \tau^{*\prime}_{yy}$	$\tau_{yy} = \bar{\tau}_{yy} + \tau'_{yy}$	$\bar{p}^*_o$
Density of the gas			
(12)	$\rho^* = \bar{\rho}^* + \rho^{*\prime}$	$\rho(y) + r(y)e^{i\alpha(x - ct)}$	$\bar{\rho}^*_o$
Pressure of the gas			
(13)	$p^* = \bar{p}^* + p^{*\prime}$	$p(y) + \pi(y)e^{i\alpha(x - ct)}$	$\bar{p}^*_o$
Temperature of the gas			
(14)	$T^* = \bar{T}^* + T^{*\prime}$	$T(y) + \theta(y)e^{i\alpha(x - ct)}$	$\bar{T}^*_o$
Coefficients of viscosity of the gas			
(15)	$\mu^*_1 = \bar{\mu}^*_1 + \mu^{*\prime}_1$	$\mu_1(y) + m_1(y)e^{i\alpha(x - ct)}$	$\bar{\mu}^*_{1o}$
(16)	$\mu^*_2 = \bar{\mu}^*_2 + \mu^{*\prime}_2$	$\mu_2(y) + m_2(y)e^{i\alpha(x - ct)}$	$\bar{\mu}^*_{1o}$

	Dimensional quantities	Dimensionless quantities	Characteristic measure
Thermal conductivity			
(18)	$k^* = \bar{k}^* + k^{*\prime}$	$\frac{1}{\sigma}\mu_1(y) + \frac{1}{\sigma_o}k(y)e^{i\alpha(x - ct)}$	$C_p\,\bar{\mu}^*_{1o}$
Wave number of the disturbance			
(19)	$\alpha^* = 2\pi/\lambda^*$	$\alpha = 2\pi/\lambda$	l^{-1}
Phase velocity of the disturbance			
(20)	c^*	c	$\bar{u}^*_o$
Specific heat at constant volume			
(21)	C_v	1	C_v
Specific heat at constant pressure			
(22)	C_p	γ	C_v
Gas constant per gram			
(23)	R^*	$\gamma - 1$	C_v
Acceleration due to gravity (in the negative direction of the y-axis)			
(24)	g	$\frac{1}{F^2}$	$\bar{u}^{*2}_o/l$

Dimensionless quantities

Froude number

(25) $$F = \frac{\bar{u}_o^*}{\sqrt{gl}}$$

Reynolds number

(26) $$R = \bar{\rho}_o^* \bar{u}_o^* l / \bar{\mu}_{1o}^*$$

Mach number

(27) $$M = \bar{u}_o^* \Big/ \sqrt{\gamma R^* \bar{T}_o^*}$$

Prandtl number

(28) $$\sigma = C_p \bar{\mu}_1^* / \bar{k}^*$$

Remarks: For the case of the boundary layer, the boundary-layer thickness δ will in general be taken to be the characteristic length; for some purposes, the displacement thickness δ_1 will be used. A bar over a quantity denotes average value, a dash denotes fluctuation; and the subscript $(\ )_o$ denotes free stream value in the case of the boundary layer. The subscripts $_r$ and $_i$ denote the real and imaginary parts of a quantity, respectively.

I - GENERAL THEORY

1. The General Equations of Disturbance

The general equations of disturbance for a perfect gas which is flowing parallel or nearly parallel to a given direction will now be derived. As has been explained, only two-dimensional motions with two-dimensional disturbances will be considered.

With the system of notation explained above, and with positive y-axis pointing vertically upward, the general equations for two-dimensional motion of a perfect gas may be written as follows:

(a) Equations of motion,

$$\frac{\partial u^*}{\partial t^*} + u^* \frac{\partial u^*}{\partial x^*} + v^* \frac{\partial u^*}{\partial y^*} = \frac{1}{\rho^*}\left(\frac{\partial \tau^*_{xx}}{\partial x^*} + \frac{\partial \tau^*_{xy}}{\partial y^*}\right) \tag{1}$$

$$\frac{\partial v^*}{\partial t^*} + u^* \frac{\partial v^*}{\partial x^*} + v^* \frac{\partial v^*}{\partial y^*} = \frac{1}{\rho^*}\left(\frac{\partial \tau^*_{xy}}{\partial x^*} + \frac{\partial \tau^*_{yy}}{\partial y^*}\right) - g^* \tag{2}$$

(b) Equation of continuity,

$$\frac{\partial \rho^*}{\partial t^*} + \frac{\partial}{\partial x^*}(\rho^* u^*) + \frac{\partial}{\partial y^*}(\rho^* v^*) = 0 \tag{3}$$

(c) Equation of energy,

$$\rho^* C_v \left\{\frac{\partial T^*}{\partial t^*} + u^* \frac{\partial T^*}{\partial x^*} + v^* \frac{\partial T^*}{\partial y^*}\right\} = (\tau^*_{xx}\,\epsilon^*_{xx} + 2\tau^*_{xy}\,\epsilon^*_{xy} + \tau^*_{yy}\,\epsilon^*_{yy})$$

$$+ \frac{\partial}{\partial x^*}\left(k^* \frac{\partial T^*}{\partial x^*}\right) + \frac{\partial}{\partial y^*}\left(k^* \frac{\partial T^*}{\partial y^*}\right) \tag{4}$$

(d) Equation of state,

$$p^* = \rho^* R^* T^* \tag{5}$$

In these equations, ϵ^*_{xx}, ϵ^*_{xy}, ϵ^*_{yy} are the components of the rate-of-strain tensor, and τ^*_{xx}, τ^*_{xy}, τ^*_{yy} are the components of the stress tensor. They are defined as follows:

$$\epsilon^*_{xx} = \frac{\partial u^*}{\partial x^*}, \qquad \epsilon^*_{xy} = \frac{1}{2}\left(\frac{\partial u^*}{\partial y^*} + \frac{\partial v^*}{\partial x^*}\right), \qquad \epsilon^*_{yy} = \frac{\partial v^*}{\partial y^*} \tag{6}$$

$$\left.\begin{aligned}
\tau^*_{xx} &= -p^* + 2\mu^*_1 \frac{\partial u^*}{\partial x^*} + \frac{2}{3}(\mu^*_2 - \mu^*_1)\left(\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*}\right) \\
\tau^*_{xy} &= \mu^*_1\left(\frac{\partial u^*}{\partial y^*} + \frac{\partial v^*}{\partial x^*}\right) \\
\tau^*_{yy} &= -p^* + 2\mu^*_1 \frac{\partial v^*}{\partial y^*} + \frac{2}{3}(\mu^*_2 - \mu^*_1)\left(\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*}\right)
\end{aligned}\right\} \tag{7}$$

The coefficients of viscosity μ^*_1 and μ^*_2 and the coefficient of heat conductivity k^*_1 are essentially functions of temperature. Hence, there results a system of five differential equations for the five variables ρ^*, T^*, p^*, u^*, v^*.

Consider a motion which is slightly disturbed from a steady state. Then it is convenient to separate any quantity $Q^*(x^*, y^*, t^*)$ into a steady-state part $\bar{Q}^*(x^*, y^*)$, and a small disturbance $Q^{*\prime}(x^*, y^*, t^*)$

$$Q^*(x^*, y^*, t^*) = \bar{Q}^*(x^*, y^*) + Q^{*\prime}(x^*, y^*, t^*) \tag{8}$$

By substituting expressions of the type (8) for each of the variables into (1) to (7), remembering that the steady-state parts satisfy those equations by themselves, and, finally, neglecting terms quadratic in the small disturbance, the following system of equations of disturbance is arrived at:

$$\frac{\partial u^{*\prime}}{\partial t^*} + \left(u^{*\prime}\frac{\partial \bar{u}^*}{\partial x^*} + v^{*\prime}\frac{\partial \bar{u}^*}{\partial y^*}\right) + \left(\bar{u}^*\frac{\partial u^{*\prime}}{\partial x^*} + \bar{v}^*\frac{\partial u^{*\prime}}{\partial y^*}\right)$$

$$= \frac{1}{\bar{\rho}^*}\left(\frac{\partial \tau^{*\prime}_{xx}}{\partial x^*} + \frac{\partial \tau^{*\prime}_{xy}}{\partial y^*}\right) - \frac{\rho^{*\prime}}{\bar{\rho}^{*2}}\left(\frac{\partial \bar{\tau}^*_{xx}}{\partial x^*} + \frac{\partial \bar{\tau}^*_{xy}}{\partial y^*}\right) \tag{9}$$

$$\frac{\partial v^{*\prime}}{\partial t^{*}} + \left(u^{*\prime}\frac{\partial \bar{v}^{*}}{\partial x^{*}} + v^{*\prime}\frac{\partial \bar{v}^{*}}{\partial y^{*}}\right) + \left(\bar{u}^{*}\frac{\partial v^{*\prime}}{\partial x^{*}} + \bar{v}^{*}\frac{\partial v^{*\prime}}{\partial y^{*}}\right)$$

$$= \frac{1}{\bar{\rho}^{*}}\left(\frac{\partial \tau_{xy}^{*\prime}}{\partial x^{*}} + \frac{\partial \tau_{yy}^{*\prime}}{\partial y^{*}}\right) - \frac{\rho^{*\prime}}{\bar{\rho}^{*2}}\left(\frac{\partial \bar{\tau}_{xy}^{*}}{\partial x^{*}} + \frac{\partial \bar{\tau}_{yy}^{*}}{\partial y^{*}}\right) \tag{10}$$

$$\frac{\partial \rho^{*\prime}}{\partial t^{*}} + \frac{\partial}{\partial x^{*}}\left(\rho^{*\prime}\bar{u}^{*} + \bar{\rho}^{*}u^{*\prime}\right) + \frac{\partial}{\partial y^{*}}\left(\rho^{*\prime}\bar{v}^{*} + \bar{\rho}^{*}v^{*\prime}\right) = 0 \tag{11}$$

$$C_v\bar{\rho}^{*}\left(\frac{\partial T^{*\prime}}{\partial t^{*}} + \bar{u}^{*}\frac{\partial T^{*\prime}}{\partial x^{*}} + \bar{v}^{*}\frac{\partial T^{*\prime}}{\partial y^{*}} + u^{*\prime}\frac{\partial \bar{T}^{*}}{\partial x^{*}} + v^{*\prime}\frac{\partial \bar{T}^{*}}{\partial y^{*}}\right)$$

$$+ C_v\rho^{*\prime}\left(\frac{\partial \bar{T}^{*}}{\partial t^{*}} + \bar{u}^{*}\frac{\partial \bar{T}^{*}}{\partial x^{*}} + \bar{v}^{*}\frac{\partial \bar{T}^{*}}{\partial y^{*}}\right) = \frac{\partial}{\partial x^{*}}\left(\bar{k}^{*}\frac{\partial T^{*\prime}}{\partial x^{*}} + k^{*\prime}\frac{\partial \bar{T}^{*}}{\partial x^{*}}\right)$$

$$+ \frac{\partial}{\partial y^{*}}\left(\bar{k}^{*}\frac{\partial T^{*\prime}}{\partial y^{*}} + k^{*\prime}\frac{\partial \bar{T}^{*}}{\partial y^{*}}\right) + \left(\bar{\tau}_{xx}^{*}\epsilon_{xx}^{*\prime} + 2\bar{\tau}_{xy}^{*}\epsilon_{xy}^{*\prime} + \bar{\tau}_{yy}^{*}\epsilon_{yy}^{*\prime}\right)$$

$$+ \left(\tau_{xx}^{*\prime}\bar{\epsilon}_{xx}^{*} + 2\tau_{xy}^{*\prime}\bar{\epsilon}_{xy}^{*} + \tau_{yy}^{*\prime}\bar{\epsilon}_{yy}^{*}\right) \tag{12}$$

$$\frac{p^{*\prime}}{\bar{p}^{*}} = \frac{\rho^{*\prime}}{\bar{\rho}^{*}} + \frac{T^{*\prime}}{\bar{T}^{*}} \tag{13}$$

where

$$\epsilon_{xx}^{*\prime} = \frac{\partial u^{*\prime}}{\partial x^*}, \qquad \epsilon_{xy}^{*\prime} = \frac{1}{2}\left(\frac{\partial u^{*\prime}}{\partial y^*} + \frac{\partial v^{*\prime}}{\partial x^*}\right), \qquad \epsilon_{yy}^{*\prime} = \frac{\partial v^{*\prime}}{\partial y^*} \tag{14}$$

and

$$\left.\begin{aligned}
\tau_{xx}^{*\prime} &= -p^{*\prime} + 2\left(\bar{\mu}_1^* \frac{\partial u^{*\prime}}{\partial x^*} + \mu_1^{*\prime} \frac{\partial \bar{u}^*}{\partial x^*}\right) + \frac{2}{3}\left\{\left(\bar{\mu}_2^* - \bar{\mu}_1^*\right)\left(\frac{\partial u^{*\prime}}{\partial x^*} + \frac{\partial v^{*\prime}}{\partial y^*}\right)\right. \\
&\quad \left. + \left(\mu_2^{*\prime} - \mu_1^{*\prime}\right)\left(\frac{\partial \bar{u}^*}{\partial x^*} + \frac{\partial \bar{v}^*}{\partial y^*}\right)\right\}, \\
\tau_{xy}^{*\prime} &= \bar{\mu}_1^*\left(\frac{\partial u^{*\prime}}{\partial y^*} + \frac{\partial v^{*\prime}}{\partial x^*}\right) + \mu_1^{*\prime}\left(\frac{\partial \bar{u}^*}{\partial y^*} + \frac{\partial \bar{v}^*}{\partial x^*}\right) \\
\tau_{yy}^{*\prime} &= -p^{*\prime} + 2\left(\bar{\mu}_1^* \frac{\partial v^{*\prime}}{\partial y^*} + \mu_1^{*\prime} \frac{\partial \bar{v}^*}{\partial y^*}\right) + \frac{2}{3}\left\{\left(\bar{\mu}_2^* - \bar{\mu}_1^*\right)\left(\frac{\partial u^{*\prime}}{\partial x^*} + \frac{\partial v^{*\prime}}{\partial y^*}\right)\right. \\
&\quad \left. + \left(\mu_2^{*\prime} - \mu_1^{*\prime}\right)\left(\frac{\partial \bar{u}^*}{\partial x^*} + \frac{\partial \bar{v}^*}{\partial y^*}\right)\right\}
\end{aligned}\right\} \tag{15}$$

Further, this investigation will be restricted to flows which are parallel or nearly parallel. By nearly parallel flows is meant flows where the boundary-layer approximation is applicable. Thus, if the main flow is nearly parallel to the x^*-axis,

$$\bar{v}^* \ll \bar{u}^*, \qquad \frac{\partial \bar{Q}^*}{\partial x^*} \ll \frac{\partial \bar{Q}^*}{\partial y^*} \tag{16}$$

(Relations of this type, of course, do not hold for the disturbances.) By considering only a local region, say around $x^* = x^*_o$, and introducing the boundary-layer approximation, the flow is regarded as essentially parallel, with every mean quantity $\bar{Q}^*$ evaluated at $x^* = x^*_o$. Thus, for parallel or nearly parallel flows, the differential equations of disturbance do not contain x^* and t^* explicitly and an attempt may be made to find solutions of the type

$$Q^{*\prime}(x^*, y^*, t^*) = q^*(y^*)e^{i\alpha^*(x^*-c^*t^*)} \tag{17}$$

Indeed, every quantity will be reduced to a dimensionless form in accordance with the scheme listed (List of Symbols); for example,

$$\bar{u}^*(y^*) = \bar{u}^*_o\, w(y), \qquad u^{*\prime} = \bar{u}^*_o\, f(y)e^{i\alpha(x-ct)} \tag{18}$$

For the present, the additional restriction of uniform free stream velocity $\bar{u}^*_o$, temperature $\bar{T}^*_o$, and so forth, will be retained. Then the <u>final dimensionless and linearized equations for the amplitudes of small disturbances in a parallel or nearly parallel main flow are as follows:</u>

$$\alpha\rho\left\{i(w-c)f + w'\varphi\right\} = -\frac{i\alpha\pi}{\gamma M^2} + \frac{\mu_1}{R}\left\{f'' + \alpha^2(i\varphi' - 2f)\right\}$$

$$+\frac{2}{3}\frac{\mu_2-\mu_1}{R}\alpha^2\left\{-f + i\varphi'\right\}$$

$$+\frac{1}{R}\left\{m_1 w'' + m_1' w' + \mu_1'(f' + i\alpha^2\varphi)\right\} \tag{19}$$

$$\alpha^2\rho\left\{i(w - c)\varphi\right\} = -\frac{\pi'}{\gamma M^2} - \frac{r}{F^2} + \frac{\mu_1\alpha}{R}\left\{2\varphi'' - if' - \alpha^2\varphi\right\}$$

$$+ \frac{2\alpha}{3}\,\frac{\mu_2 - \mu_1}{R}\left\{\varphi'' + if'\right\}$$

$$+ \frac{\alpha}{R}\left\{im_1w' + 2\mu_1'\varphi' + \frac{2}{3}(\mu_2 - \mu_1)(\varphi' + if)\right\} \tag{20}$$

$$i(w - c)r + \rho(\varphi' + if) + \rho'\varphi = 0 \tag{21}$$

$$\alpha\rho\left\{i(w - c)\theta + T'\varphi\right\} = -\alpha(\gamma - 1)\rho T(\varphi' + if)$$

$$+ \frac{\gamma}{R\sigma_o}\left\{\mu_1(\theta'' - \alpha^2\theta) + (m_1T')' + \mu_1'\theta'\right\}$$

$$+ \gamma\,\frac{(\gamma - 1)}{R}\,M^2\left\{m_1w'^2 + 2\mu_1w'(f' + i\alpha^2\varphi)\right\} \tag{22}$$

$$\frac{\pi}{p} = \frac{r}{\rho} + \frac{\theta}{T} \tag{23}$$

The following two dimensionless equations for mean quantities should also be noted:

Equation of state,

$$p = \rho T \tag{24}$$

Equation of static pressure gradient across the boundary layer,

$$p' = \frac{\gamma M^2}{F^2} \rho \tag{25}$$

In all these equations, and in all subsequent equations, a dash denotes differentiation with respect to the dimensionless variable y, and should not be connected with the idea of a fluctuation. For example,

$$\mu_1' = \frac{d\mu_1}{dy} = \frac{d\mu_1}{dT} \frac{dT}{dy} \tag{26}$$

while the fluctuation m_1 is given by

$$m_1 = \theta \frac{d\mu_1}{dT} \tag{27}$$

Note that in (18) a characteristic velocity has been used as the reference variable. This stresses the role of the inertial forces. Indeed, the simultaneous comparison of the inertia forces with pressure and gravitation, as embodied in the Mach number

$$M = \frac{\bar{u}_o^*}{c_o^*}$$

and the Froude number

$$F = \frac{\bar{u}_o^*}{\sqrt{g^* l}}$$

makes it difficult to consider the limit of small inertial forces $\bar{u}_0^* \to 0$. For in such a limiting case, the phenomenon is essentially governed by pressure and gravitational forces, which both become infinitely large compared with the reference inertial force. In aerodynamical problems, however, this limiting case is not of importance.

Mathematically speaking, a singularity is brought into (20) if the neighborhood of $F = 0$ is considered. Thus, it is possible to study only the case of a small Mach number. The limit of vanishing Mach number and vanishing Froude number can be considered only when $r = 0$. These statements will become clearer after reading the detailed discussions in section 3.

To study the case of an extremely large Mach number, on the other hand, it would be more convenient to use the stagnation or "rest" values of pressure, density, and temperature as characteristic measures, rather than the free stream values.

2. Analytical Nature of the System of Equations of Disturbance and Its Solutions

The system of equations of disturbance (19) to (23) consists of five linear equations in the five variables $f, \varphi, \pi, r, \theta$, with p, ρ, T, w supposedly known from the steady-state solutions. Before applying this system to any definite problem, it is necessary to know clearly its analytical nature; for example, the number of sets of linearly independent solutions it possesses must be known. It is also desirable to know the general analytical nature of the solutions in the variable y and in the parameters M^2, F^2, R, α, and c. In all these discussions of analytical nature, both the variable y and the parameters will be regarded as complex.

To settle these questions, it is convenient to choose a number of new variables $Z_1, \ldots, Z_n$ and rewrite the system into the form

$$\frac{dZ_i}{dy} = \sum_{j=1}^{n} A_{ij}(y)\, Z_j \quad (i = 1, 2, \ldots, n) \tag{28}$$

where A_{ij} are known functions of y. Since (19), (20), and (22) involve the second derivatives of f, φ, and θ, it seems desirable to choose the six dependent variables as f, φ, θ, f', φ', θ'. In this way, the equations can be set equivalent to six equations of the type (28), if r and π are supposedly solved algebraically from (21) and (23). It is seen, therefore, that the system of five equations (19) to (23) is actually equivalent to six homogeneous linear differential equations of the first order, and there are six linearly independent solutions.

However, this choice of the dependent variables is not satisfactory. It leads at once to the suspicion that the solutions have singularities at the point where $w = c$. For, in solving for r from (21) a singularity is introduced into the coefficients $A_{ij}(y)$ of the system (28). Physically, the solutions cannot have such a singularity for real values of y. Hence, it is necessary that such a singularity be only apparent. Indeed, this can be shown to be true by a new choice of the dependent variables.

To be more precise, let it be assumed that the known functions w, p, ρ, T are analytic functions of y and of the parameters M^2, $\frac{1}{F^2}$. These functions may be regarded as independent of the Reynolds number R, when the characteristic length l is properly chosen. This assumption is related to the boundary-layer approximation, and is therefore accurate up to the same order. For example, for the Blasius profile, it is accurate up to the order of $(R\delta)^{-1/2}$, δ being the thickness of the boundary layer.

Now choose the system of dependent variables

$$\left.\begin{aligned} &Z_1 = f, \qquad Z_2 = f', \qquad Z_3 = \varphi \\ &Z_4 = \frac{\pi}{M^2} \qquad Z_5 = \theta, \qquad Z_6 = \theta' \end{aligned}\right\} \tag{29}$$

Then, at once,

$$\frac{dZ_1}{dy} = Z_2 \tag{30}$$

$$\frac{dZ_5}{dy} = Z_6 \tag{31}$$

The equation (23) becomes

$$r = \frac{\rho}{p} M^2 Z_4 - \frac{\rho}{T} Z_5 \tag{32}$$

which makes it possible to eliminate the variable r without increasing the order of the differential equations and without introducing any singularity. Then from (19), (21), and (23), solve for Z'_2, Z'_3, and Z'_6, which, when reduced with the help of (30) to (32) are equations of the type (28) with $A_{ij}(y)$ regular in both y and the parameters. The equation for Z_3 is

$$\frac{dZ_3}{dy} = -iZ_1 - \frac{\rho'}{\rho} Z_3 - i(w - c)\left\{\frac{M^2}{p} Z_4 + \frac{1}{T} Z_5\right\} \tag{33}$$

but the other two equations are too lengthy to be written out explicitly. They are of the following general nature:

$$\frac{dZ_2}{dy} = \frac{\alpha R}{\mu_1} \left\{\rho\left[i(w - c)Z_1 + w'Z_3\right] + iZ_4\right\} + O(1) \tag{34}$$

$$\frac{dZ_6}{dy} = \frac{\sigma\alpha R}{\gamma\mu_1} \left\{\gamma\rho\left[i(w - c)Z_5 + T'Z_3\right]\right.$$

$$\left. - (\gamma - 1)\left[p'Z_3 - i(w - c)M^2 Z_4\right]\right\} + O(1) \tag{35}$$

where $O(1)$ denotes a linear function of $Z_1, \ldots, Z_6$ which is of the order of unity in the parameter R and is regular in the parameter M^2.

The differential equation for Z'_4 must be obtained from (20) in a slightly different manner. It is necessary first to eliminate φ'' by using (33) and then solve for $\pi' = M^2Z'_4$. There is obtained

$$\frac{dZ_4}{dy} = \left\{1 + \gamma M^2 \frac{i\alpha}{R}\frac{2}{3}(\mu_2 + 2\mu_1)\frac{(w - c)}{p}\right\}^{-1}$$

$$\times \left\{-\frac{1}{F^2}\left(\frac{\rho}{p}M^2Z_4 - \frac{\rho}{T}Z_5\right) - i\alpha(w - c)\rho Z_3 + \frac{1}{R}O(1)\right\} \tag{36}$$

where O(1) has the same general meaning as before. It is noted that the last step is the only division involved in this process of elimination. Thus, unless

$$1 + \gamma M^2 \frac{i\alpha}{R}\frac{2}{3}(\mu_2 + 2\mu_1)\frac{w - c}{p} = 0 \tag{37}$$

(which is not possible for $|R| \gg 1$), the system of differential equations (30) to (36) is regular in y and in the parameters. But since the regularity breaks down for infinite R and for R satisfying (37), any expansion of the solution as a power series in R must be in the form of Laurent series. In the parameters $\frac{1}{F^2}$ and M^2, the coefficients are entire functions; in the parameters α and c, they are analytic in a region including the origin.

From the general existence proof of the solutions of linear differential equations by means of successive approximations, it is clear that these properties of the coefficients persist in the solutions. That is, there exists a fundamental system of six solutions $Z_i(y;\ M^2,\ 1/F^2,\ R,\ \alpha,\ c)$ $(i = 1, 2, 3, 4, 5, 6)$ which are analytic functions of y and of the parameters.

Now consider a few limiting cases: (1) $M^2 \to 0$, (2) $1/F^2 \to 0$, (3) $R \to \infty$. As discussed at the end of section 2, if $M^2 \to 0$ by making the velocity $\bar{u}_o^* \longrightarrow 0$, then F^2 and R approach zero at the same time. This is certainly not what the authors wish to discuss. Rather, they are thinking of the solutions Z_1 as expanded in power series of the parameters, say,

$$Z_1\left(y;\ M^2,\ \frac{1}{F^2},\ R,\ \alpha,\ c\right) = Z_1^{(o)}\left(y;\ \frac{1}{F^2},\ R,\ \alpha,\ c\right)$$

$$+ M^2 Z_1^{(1)}\left(y;\ \frac{1}{F^2},\ R,\ \alpha,\ c\right) + \ldots \tag{38}$$

and retaining only the zeroth order term as an approximation. This process is valid so long as M^2 is sufficiently small compared with unity, while both F^2 and R are of their usual magnitudes (namely, much larger than unity). For convenience, the mathematical process $M^2 \longrightarrow 0$ will still be preserved. But this must not be confused with any physical requirement that $\bar{u}_o^* \longrightarrow 0$ or $\bar{c}_o^* \longrightarrow \infty$, the latter being in contradiction with the equation of state. The limiting case $R \longrightarrow \infty$ is an asymptotic approximation and will be dealt with more carefully below.

Case (1) $M^2 \longrightarrow 0$. With the relation (24) in mind, the equations (19) to (23) become $(\omega = \pi/\gamma M^2)$

$$\alpha\rho\left\{i(w - c)f + w'\varphi\right\} = i\alpha\omega + \frac{\mu_1}{R}\left\{f'' + \alpha^2(i\varphi' - 2f)\right\}$$

$$+ \frac{2}{3}\,\frac{\mu_2 - \mu_1}{R}\left\{-f + i\varphi'\right\}\alpha^2 + \frac{1}{R}\left\{m_1 w'' + m_1' w' + \mu_1'(f' + i\alpha^2\varphi)\right\} \tag{39}$$

$$\alpha\rho\left\{i(w-c)\varphi\right\} = -\omega' - \frac{r}{F^2} + \frac{\mu_1\alpha}{R}\left\{2\varphi'' - if' - \alpha^2\varphi\right\} + \frac{2\alpha}{3}\,\frac{\mu_2 - \mu_1}{R}\left\{\varphi'' + if'\right\}$$

$$+\frac{\alpha}{R}\left\{im_1w' + 2\mu_1'\varphi' + \frac{2}{3}(\mu_2 - \mu_1)(\varphi' + if)\right\} \tag{40}$$

$$i(w-c)r + \rho(\varphi' + if) + \rho'\varphi = 0 \tag{41}$$

$$\alpha\rho\left\{i(w-c)\theta + T'\varphi\right\} = -\alpha(\gamma-1)\rho T(\varphi' + if)$$

$$+\frac{\gamma}{R\sigma_0}\left\{\mu_1(\theta'' - \alpha^2\theta) + (m_1T')' + \mu_1'\theta'\right\} \tag{42}$$

$$0 = \frac{r}{\rho} + \frac{\theta}{T} \tag{43}$$

The fact that equation (25) reduces to $p' = 0$ indicates that the gravitational force is important in these problems only insofar as the buoyancy corresponding to density fluctuations is concerned and not in connection with the determination of mean density distribution.

This set of equations (39) to (43) is different from that used by Schlichting (reference 7), who neglected temperature variations but included density variations. In his case, (42) yields the condition of incompressibility

$$\varphi' + if = 0 \tag{42a}$$

and (41) becomes

$$i(w-c)r + \rho'\varphi = 0 \tag{41a}$$

He also made certain other minor reductions in (39) and (40). The justificatior of Schlichting's assumptions is not obvious. Also, his <u>complete</u> equation of disturbance (equation (11), p. 319, reference 7) has a singularity at the point where the phase velocity is equal to the mean velocity of the flow. This gives rise to multiple-valued solutions, to which it is difficult to assign a proper physical interpretation.

<u>Reduction to the equation of Orr and Sommerfeld for an homogeneous incompressible fluid</u>.- This simple case is obtained from the limiting case of zero Mach number with the additional requirements that the mean pressure, mean temperature, and mean density are constants. These conditions can hold only in the case where there is no conduction of heat across the boundaries. Otherwise, there must be a finite temperature gradient at the boundary. When p, ρ, T are constants, indeed $p = \rho = T = 1$; then the equations (41) to (43) give

$$i(w - c)r + (\varphi' + if) = 0 \tag{44}$$

$$r + \theta = 0 \tag{45}$$

$$i\alpha(w - c)\theta = \frac{1}{R\sigma_o}\mu_{1o}(\theta'' - \alpha^2\theta) \tag{46}$$

Multiplying (46) by θ, adding the corresponding complex conjugate, and integrating between the boundaries along the real axis of the y-plane gives

$$-2\int_{y_1}^{y_2}|\theta'|^2 dy - 2(\alpha^2 + \alpha R\sigma_o c_i/\mu_{1o})\int_{y_1}^{y_2}|\theta|^2 dy = 0$$

if the boundary conditions are $\theta' = 0$ at both boundaries. Thus the solution of (46) is not identically zero only when

$$c_i \leq -\alpha\mu_{1o}/\sigma_o R \tag{47}$$

Thus, if the main interest is in the limit of stability (c_i changing sign), only the solution $\theta = 0$ need be considered. Then $r = 0$ by (45), and the equation (44) reduces to

$$\varphi' + if = 0 \tag{44a}$$

The equations (19) and (20) then become

$$\left.\begin{aligned} i\alpha(w - c)f + \alpha w'\varphi &= -i\alpha\varpi + \frac{1}{R}(f'' - \alpha^2 f) \\ i\alpha^2(w - c)\varphi &= -\varpi' + \frac{1}{R}(\varphi'' - \alpha^2\varphi) \end{aligned}\right\} \tag{48}$$

Eliminating ϖ from these equations and then substituting f from (44a), the equation of Orr and Sommerfeld is obtained.

Case (2) $F^2 \rightarrow \infty$. In this case, all the equations (19) to (25) remain unaltered, except that the terms in $1/F^2$ should be dropped from (20) and (25). This is the case which will be discussed more in detail.

Case (3) $R \rightarrow \infty$. In this case, the equations (19) to (23) become

$$\rho\left\{i(w - c)f + w'\varphi\right\} = -\frac{i\pi}{\gamma M^2} \tag{49}$$

$$\rho\left\{i\alpha^2(w - c)\varphi\right\} = -\frac{\pi'}{\gamma M^2} - \frac{r}{F^2} \tag{50}$$

$$i(w - c)r + \rho(\varphi' + if) + \rho'\varphi = 0 \tag{51}$$

$$\rho\left\{i(w - c)\,\theta + T'\varphi\right\} = -(\gamma - 1)\rho T(\varphi' + if) \tag{52}$$

$$\frac{\pi}{p} = \frac{r}{\rho} + \frac{\theta}{T} \tag{53}$$

The equations (24), (25) remain unchanged. It is to be noticed that the orders of the differential equations are reduced. This is consistent with the fact that the solutions have an essential singularity at $R = \infty$.

After the elimination of f, π, r, θ, the final differential equation for φ reads

$$\frac{d}{dy}\left\{\frac{(w - c)\varphi' - w'\varphi}{T - M^2(w - c)^2}\right\} - \frac{\alpha^2(w - c)}{T}\varphi = \frac{1}{F^2}\,\underline{L}(\varphi) \tag{54}$$

where $\underline{L}(\varphi)$ is a linear expression in φ involving φ' and φ.

<u>The boundary-value problems.</u>- For a given physical problem, there are usually associated certain boundary conditions on the disturbance. For example, for flow between fixed parallel plates, the velocity disturbances must vanish at these plates. Also, if these plates are insulators, the temperature gradient must be zero. <u>In general</u>, therefore, it may be expected that six boundary conditions will be satisfied. Since there are <u>six</u> homogeneous linear differential equations in <u>six</u> variables, there is a characteristic-value problem if the boundary conditions are also homogeneous. Let

$$Z_i = Z_{ij}\left(y;\ M^2,\ \frac{1}{F^2},\ R,\ \alpha,\ c\right),\quad i,\ j = 1,\ 2,\ .\ .\ .\ ,\ 6 \tag{55}$$

represent a complete system of <u>six</u> solutions, and let the boundary conditions be

$$L_k\left\{Z_1,\ Z_2,\ .\ .\ .\ ,\ Z_6\right\} = 0 \text{ at } y = y_k, \quad k = 1,\ 2,\ .\ .\ .\ ,\ 6 \tag{56}$$

where L_k is a homogeneous linear function. Then if the solution is

$$Z_i = \sum_{j=1}^{6} A_j Z_{ij}(y) \qquad i = 1,\ 2,\ .\ .\ .\ ,\ 6 \tag{57}$$

there results

$$\left.\begin{aligned} &\sum_{j=1}^{6} A_j L_k\left\{Z_{1j}(y_k),\ Z_{2j}(y_k),\ .\ .\ .\ ,\ Z_{6j}(y_k)\right\} = 0 \\ &\qquad k = 1,\ 2,\ .\ .\ .\ ,\ 6 \end{aligned}\right\} \tag{58}$$

Hence, there follows the secular equation

$$\left| L_{jk}\left(M^2, \frac{1}{F^2},\ R,\ \alpha,\ c\right) \right| = 0 \tag{59}$$

where

$$\left.\begin{aligned} &L_{jk} = L_k\left\{Z_{1j}(y_k),\ Z_{2j}(y_k),\ .\ .\ .\ Z_{6j}(y_k)\right\}, \\ &\qquad j,\ k = 1,\ 2,\ .\ .\ .\ ,\ 6 \end{aligned}\right\} \tag{60}$$

If equation (59) can be solved for c, there results

$$c = c\left(\alpha, \quad R, \quad M^2, \quad \frac{1}{F^2}\right) \tag{61}$$

For real values of α, R, M^2 and $1/F^2$, it is convenient to split (61) into its real and imaginary parts,

$$c_r = c_r\left(\alpha, \quad R, \quad M^2, \quad \frac{1}{F^2}\right) \tag{62}$$

$$c_i = c_i\left(\alpha, \quad R, \quad M^2, \quad \frac{1}{F^2}\right) \tag{63}$$

The condition $c_i = 0$ gives the limit of stability.

For incompressible fluids without the effect of gravity, plot the curve $c_i(\alpha,R) = 0$ in the α-R plane. Here, it has to be done for a series of values of M^2 and $1/F^2$.

Continuous characteristic values.- In case one of the conditions (56) is absent (cf. the case of "supersonic disturbances" in a boundary layer, sec. 5), no such relation as (61) exists, and a solution satisfying the remaining five boundary conditions (and certain other conditions of boundedness) can always be found. This is the case of "continuous characteristic values." The physical significance of such solutions will be discussed as the case turns up.

3. Solution of the System of Differential Equations by Method of Successive Approximations

The exact solution of the system of differential equations (19) to (23) or rather (30), (31), and (33) to (36) is almost impossible. With the appearance of the small parameter $1/\alpha R$ it seems desirable to use the method of successive approximations. The general plan of solution will be exactly the same as in the case of an incompressible fluid (reference 1). Two methods of solution are possible, the first using convergent series and the second using asymptotic series.

(1) Solution by means of convergent series.- In the first method, introduce the parameter

$$\epsilon = (\alpha R)^{-\frac{1}{3}} \tag{64}$$

and the new variable

$$\eta = (y - y_c)/\epsilon \qquad \text{where} \qquad w(y_c) = c \tag{65}$$

The equations (30), (31), and (33) to (36) then take the following forms:

$$\frac{dZ_1}{d\eta} = \epsilon Z_2 \tag{66}$$

$$\frac{dZ_5}{d\eta} = \epsilon Z_6 \tag{67}$$

$$\frac{dZ_3}{d\eta} = -i\epsilon Z_1 - \frac{\rho'}{\rho}\epsilon Z_3 - i\epsilon(w - c)\left\{\frac{M^2}{p} Z_4 + \frac{1}{T} Z_5\right\} \tag{68}$$

$$\frac{dZ_2}{d\eta} = \frac{1}{\epsilon^2 \mu_1}\left\{\rho\left[i(w - c)Z_1 + w' Z_3\right] + iZ_4\right\} + \epsilon O(1) \tag{69}$$

$$\frac{dZ_6}{d\eta} = \frac{\sigma}{\epsilon^2 \mu_1}\left\{ \rho\left[i(w - c)Z_5 + T'Z_3\right]\right.$$

$$\left. - \frac{\gamma - 1}{\gamma}\left[p'Z_3 - i(w - c)M^2 Z_4\right]\right\} + O(1) \tag{70}$$

$$\frac{dZ_4}{d\eta} = \epsilon \frac{-\frac{1}{F^2}\left\{\frac{\rho}{p} M^2 Z_4 - \frac{\rho}{T} Z_5\right\} - i\alpha^2(w - c)Z_3 + \alpha\epsilon^3 O(1)}{1 + i\alpha^2\epsilon^3 \frac{2}{3}(\mu_2 + 2\mu_1)\frac{w - c}{p}\gamma M^2} \tag{71}$$

The average quantities w, ρ, T, and so forth, are to be regarded as expanded in Taylor's series in the neighborhood of $w = c$; thus,

$$\left.\begin{aligned} w - c &= w_c'(\epsilon\eta) + \frac{w_c''}{2!}(\epsilon\eta)^2 + \ldots, \\ \rho &= \rho_c + \rho_c'(\epsilon\eta) + \ldots, \qquad \text{etc.} \end{aligned}\right\} \tag{72}$$

(In general, attach the subscript c to denote quantities at the critical layer where $w = c$.) The coefficients of the system of equations (66) to (71) are therefore convergent power series in ϵ so long as the power series (72) are convergent and the condition (37) is not violated. An attempt can then also be made to obtain a fundamental system of solutions as power series of ϵ. A consultation of equations (66) to (71) shows that the solutions should be of the following forms:

$$\left.\begin{aligned}
Z_1 &= X_1^{(o)}(\eta) + \epsilon X_1^{(1)}(\eta) + \ldots \\
\epsilon Z_2 &= X_2^{(o)}(\eta) + \epsilon X_2^{(1)}(\eta) + \ldots \\
\epsilon^{-1} Z_3 &= X_3^{(o)}(\eta) + \epsilon X_3^{(1)}(\eta) + \ldots \\
\epsilon^{-1} Z_4 &= X_4^{(o)}(\eta) + \epsilon X_4^{(1)}(\eta) + \ldots \\
Z_5 &= X_5^{(o)}(\eta) + \epsilon X_5^{(1)}(\eta) + \ldots \\
\epsilon Z_6 &= X_6^{(o)}(\eta) + \epsilon X_6^{(1)}(\eta) + \ldots
\end{aligned}\right\} \tag{73}$$

Now, substitute equation (73) in equations (66) to (71) and compare the coefficients of different powers of ϵ. The initial approximation gives

$$\frac{dX_1^{(o)}}{d\eta} = X_2^{(o)} \tag{74}$$

$$\frac{dX_5^{(o)}}{d\eta} = X_6^{(o)} \tag{75}$$

$$\frac{dX_3^{(o)}}{d\eta} = -iX_1^{(o)} \tag{76}$$

$$\frac{dX_2^{(o)}}{d\eta} = \frac{\rho_c}{\mu_{1c}} w_c'(i\eta X_1^{(o)} + X_3^{(o)}) + \frac{1}{\mu_{1c}} X_4^{(o)} \tag{77}$$

$$\frac{dX_6^{(o)}}{d\eta} = \frac{\sigma_c \rho_c}{\mu_{1c}} \left\{ iw_c'\eta X_5^{(o)} + T_c' X_3^{(o)} - \frac{\gamma - 1}{\gamma p_c} p_c' X_3^{(o)} \right\} \tag{78}$$

$$\frac{dX_4^{(o)}}{d\eta} = \frac{1}{F^2} \frac{p_c}{T_c} X_5^{(o)} \tag{79}$$

Dropping the superscript zero and recalling the definitions (29) and (73) gives, to the initial approximation,

$$f = X_1(\eta), \quad \varphi = \epsilon X_3(\eta), \quad \theta = X_5(\eta), \quad \frac{\pi}{M^2} = X_4(\eta) \tag{80}$$

which satisfy the equations

$$\frac{d^3X_1}{d\eta^3} - \frac{iw_c'}{\nu_{1c}}\eta\frac{dX_1}{d\eta} = \frac{i}{F^2\nu_{1c}T_c}X_5 \tag{81}$$

$$\frac{d^2X_5}{d\eta^2} - \frac{iw_c'\sigma_c}{\nu_{1c}}\eta X_5 = \frac{\alpha\sigma_c}{\nu_{1c}}\left(T_c' - \frac{\gamma - 1}{\gamma}\frac{p_c'}{\rho_c}\right)X_3 \tag{82}$$

$$\frac{dX_3}{d\eta} = - iX_1 \tag{83}$$

where ν_1 is the kinematic viscosity coefficient μ_1/ρ, and σ is the Prandtl number.

The higher approximations give nonhomogeneous equations, which are too complicated to be written out in detail. The homogeneous parts of those equations are the <u>same</u> as (74) to (79) with $X_i^{(0)}$ replaced by $X_i^{(n)}$ $(i = 1,2, \ldots,6)$. The inhomogeneous part consists of functions of lower orders and is therefore known. Thus, if the equations (74) to (79) can be solved, these equations for higher approximations can all be solved by means of quadratures.

If X_5 is eliminated from (81) by means of (82) and (83), a differential equation of the sixth order is obtained for X_1, which will give six independent solutions. The corresponding function X_3 and X_5 can then be obtained from (83) and (82). This is fairly complicated; fortunately, the case where the Froude number is very large is of interest. Thus, as an initial approximation, (81) may be reduced to

$$\frac{d^3X_1}{d\eta^3} - i\frac{w_c'}{\nu_{1c}}\eta\frac{dX_1}{d\eta} = 0 \tag{84}$$

the solutions of which are

$$X_{11} = \int H_{1/3}^{(1)}\left[\frac{2}{3}(i\zeta)^{\frac{3}{2}}\right]\zeta^{\frac{1}{2}}d\zeta, \quad X_{12} = \int H_{1/3}^{(2)}\left[\frac{2}{3}(i\zeta)^{\frac{3}{2}}\right]\zeta^{\frac{1}{2}}d\zeta, \quad X_{13} = 1 \tag{85}$$

where

$$\zeta = \left(\frac{w_c'}{\nu_{1c}}\right)^{1/3} \eta \tag{86}$$

Apparently, three solutions are missing. These can be supplied by

$$X_{14} = 0, \quad X_{15} = 0, \quad X_{16} = 0 \tag{87}$$

The reason these solutions are not trivial will be clear when the corresponding functions X_3 and X_5 are worked out below.

From (83), it is clear that the functions X_3 corresponding to (85) are

$$\left.\begin{aligned}
X_{31} &= -i\alpha\left(\frac{w_c'}{\nu_{1c}}\right)^{1/3}\left\{\zeta\int H_{1/3}^{(1)}\left[\frac{2}{3}(i\zeta)^{\frac{3}{2}}\right]\zeta^{\frac{1}{2}}d\zeta - \int H_{1/3}^{(1)}\left[\frac{2}{3}(i\zeta)^{\frac{3}{2}}\right]\zeta^{\frac{3}{2}}d\zeta\right\} \\
X_{32} &= -i\alpha\left(\frac{w_c'}{\nu_{1c}}\right)^{1/3}\left\{\zeta\int H_{1/3}^{(2)}\left[\frac{2}{3}(i\zeta)^{\frac{3}{2}}\right]\zeta^{\frac{1}{2}}d\zeta - \int H_{1/3}^{(2)}\left[\frac{2}{3}(i\zeta)^{\frac{3}{2}}\right]\zeta^{\frac{3}{2}}d\zeta\right\} \\
X_{33} &= -i\alpha\left(\frac{w_c'}{\nu_{1c}}\right)^{-\frac{1}{3}}\zeta
\end{aligned}\right\} \tag{88}$$

The functions X_3 corresponding to (87) are

$$X_{34} = 1, \quad X_{35} = 0, \quad X_{36} = 0 \tag{89}$$

Corresponding to each function X_3, there are two particular integrals X_5 obtainable from (82). These are expressible in quadratures involving Hankel functions, for the left-hand side of (82) has the solutions

$$X_{55} = \zeta^{\frac{1}{2}} H_{1/3}^{(1)}\left[\frac{2}{3}(i\zeta)^{\frac{3}{2}}\sqrt{\sigma_c}\right], \quad X_{56} = \zeta^{\frac{1}{2}} H_{1/3}^{(2)}\left[\frac{2}{3}(i\zeta)^{\frac{3}{2}}\sqrt{\sigma_c}\right] \tag{90}$$

Indeed, the sets of functions (X_{15}, X_{35}, X_{55}), (X_{16}, X_{36}, X_{56}) form two sets of solutions of (81) to (83), and it now becomes clear that (87) is not trivial.

Next, the asymptotic solutions will be studied, and the convergent solutions will be discussed later when the boundary conditions are considered.

(2) Solution by means of asymptotic series.- Analogous to the incompressible case, two asymptotic solutions are obtained by the most naïve method of expanding the solutions in powers of $(\alpha R)^{-1}$. In the present case, the initial approximation gives the inviscid equation (54), which is of the second order. The equations for successive higher approximations are inhomogeneous; the homogeneous part is of the same form as (54), while the inhomogeneous part is related to functions of lower orders. Hence, the integration of all the differential equations can be done in terms of quadratures, as soon as (54) is solved.

Four other asymptotic solutions are obtained by putting

$$Z_i = f_i \exp\left\{(\alpha R)^{\frac{1}{2}} \int g \, dy\right\} \tag{91}$$

in (30) to (36), where

$$\left.\begin{aligned} f_i &= f_i^{(0)} + f_i^{(1)}(\alpha R)^{-\frac{1}{2}} + \ldots, \quad i = 1, 3, 4, 5 \\ f_i &= f_i^{(0)}(\alpha R)^{\frac{1}{2}} + f_i^{(1)} + f_i^{(2)}(\alpha R)^{-\frac{1}{2}} + \ldots, \quad i = 2,6 \end{aligned}\right\} \tag{92}$$

while g is independent of (αR). The initial approximations are

$$\left.\begin{aligned} (Z_1,Z_3,Z_4,Z_5) &= (1,0,0,0) \exp\left\{\pm(\alpha R)^{\frac{1}{2}}\int_{y_c}^{y}\sqrt{\frac{i}{\nu_1}(w-c)}\, dy\right\} \\ (Z_2,Z_6) &= (\alpha R)^{\frac{1}{2}}\left(\pm\sqrt{\frac{i}{\nu_1}(w-c)},\ 0\right) \exp\left\{\pm(\alpha R)^{\frac{1}{2}}\int_{y_c}^{y}\sqrt{\frac{i}{\nu_1}(w-c)}\, dy\right\} \end{aligned}\right\} \tag{93}$$

$$\left.\begin{aligned} (Z_1,Z_3,Z_4,Z_5) &= (0,0,0,1) \exp\left\{\pm(\alpha R)^{\frac{1}{2}}\int_{y_c}^{y}\sqrt{\frac{i\sigma}{\nu_1}(w-c)}\, dy\right\} \\ (Z_2,Z_6) &= (\alpha R)^{\frac{1}{2}}\left(0,\pm\sqrt{\frac{i\sigma}{\nu_1}(w-c)}\right) \exp\left\{\pm(\alpha R)^{\frac{1}{2}}\int_{y_c}^{y}\sqrt{\frac{i\sigma}{\nu_1}(w-c)}\, dy\right\} \end{aligned}\right\} \tag{94}$$

Each of these sets contains two solutions. From these expressions, it appears that the solutions are multiple-valued. Actually, they are valid only for certain regions of the complex plane determined by comparing them with the asymptotic expansions of the convergent solutions (85) to (90). Analogous to the incompressible case, the asymptotic expressions hold when (cf. equation (5.4) of reference 1)

$$-\frac{7\pi}{6} < \arg(\zeta) < \frac{\pi}{6}, \quad \text{and} \quad -\frac{7\pi}{6} < \arg(\zeta\sigma_c^{1/3}) < \frac{\pi}{6} \tag{95}$$

simultaneously. If c is very close to a real number, this means that the expressions (93), (94) represent solutions in a <u>connected</u> region which contains at least a substantial portion of the real axis. This fact will be seen to be of significance in discussing the boundary-value problems.

Similar considerations hold for the solutions of the inviscid equations. These solutions appear to possess a logarithmic singularity at the point $w = c$. As in the incompressible case, the asymptotic expansions of the convergent solutions bring these solutions into correspondence with X_{33} and X_{34}, and the restrictions (95) explain this apparently multiple-valued nature of the solutions.

Analogous to the incompressible case, there are points on the real axis where the asymptotic solutions fail in the case of damped disturbances. These are interpreted as "inner viscous layers," where the effect of viscosity is not negligible no matter how large the Reynolds number is. Considering the conditions (95), it is seen that there are, in general, <u>four</u> of them. For the lines

$$\arg(\zeta) = -\frac{7\pi}{6}, \quad \arg(\zeta) = \frac{\pi}{6}$$

$$\arg(\zeta\sigma_c^{1/3}) = -\frac{7\pi}{6}, \quad \arg(\zeta\sigma_c^{1/3}) = \frac{\pi}{6}$$

intersect the real axis in four points, if σ is not a constant and $c_i < 0$. These are the points where the inviscid solutions fail. The four points reduce to the single point y_c when c is real, and there is no intersection when $c_i > 0$. Hence, the four inner viscous layers coalesce into one in the case of neutral disturbances and disappear completely for self-excited disturbances. The significance of these results for the studies in part II is also similar to that in the incompressible case.

4. Boundary-Value Problems

Having obtained the solutions in convergent series of ϵ and in asymptotic series, the boundary-value problems discussed briefly in section 3 will now be enlarged upon. The physical requirements give rise to certain mathematical conditions on the real axis of the complex y-plane. In general, the boundary conditions for the velocity disturbances are independent of the temperature disturbances. For example, for flow in a channel with walls at y_1 and y_3, the boundary conditions

$$f(y_1) = \varphi(y_1) = f(y_3) = \varphi(y_3) = 0 \tag{96}$$

must be satisfied whatever the conditions on the temperature disturbances may be. These conditions are identically satisfied by the solutions X_{15} and X_{16}, to the proper degree of approximation. Thus, the quantities L_{jk} defined by (60) vanish if $j = 5,6$ and $k = 1,2,3,4$ (say). Hence, condition (59) reduces to

$$\left| L_{jk} \right| = 0, \quad j,k = 1,2,3,4 \tag{97}$$

The characteristic-value problem therefore does not explicitly depend on the temperature disturbances in the initial approximation. Indeed, after the temperature disturbance corresponding to the characteristic oscillations has been determined, it is always possible to satisfy the boundary conditions for the temperature disturbances by including a suitable linear combination of the solutions X_{55} and X_{56}. The corresponding velocity disturbances are identically zero and will therefore not interfere with the boundary conditions imposed upon X_{55} and X_{56}. Such a situation is in general the case. The characteristic-value problem therefore becomes very similar to that in the case of the incompressible fluid. Two inviscid solutions $f_{1,2}$, $\varphi_{1,2}$ are derived from (54) and two viscous solutions from (85) and (88):

$$\left.\begin{aligned}
f_{3,4} &= \int H_{1/3}^{(1),(2)}\left[\tfrac{2}{3}(i\zeta)^{3/2}\right]\zeta^{1/2}\,d\zeta \\
\varphi_{3,4} &= -i\left(\frac{w_c'}{\nu_{1c}}\right)^{\frac{1}{3}}\left\{\zeta\int H_{1/3}^{(1),(2)}\left[\tfrac{2}{3}(i\zeta)^{\frac{3}{2}}\right]\zeta^{\frac{1}{2}}\,d\zeta - \int H_{1/3}^{(1),(2)}\left[\tfrac{2}{3}(i\zeta)^{\frac{3}{2}}\right]\zeta^{\frac{3}{2}}\,d\zeta\right\}
\end{aligned}\right\} \tag{98}$$

As explained in the last section, the asymptotic solutions hold in a connected region containing most points of the real axis except a neighborhood of the point y_c. Thus, there will be no difficulty in using the solutions $f_{1,2}$, $\varphi_{1,2}$ for the discussion of boundary-value problems. The path leading from one boundary point to another lies essentially in the lower half plane.

Now, let these results be applied to the case of the boundary layer. Begin by making a careful investigation of the boundary conditions. At the wall the conditions $f = \varphi = 0$ hold. However, f may be replaced by a linear combination of φ and φ'. In the case of the viscous solutions, $\varphi'_{3,4} + if_{3,4} = 0$, so that $f_{3,4} = i\varphi'_{3,4}$. For the inviscid solutions, the relation

$$f_{1,2} = i\frac{T\varphi'_{1,2} - M^2(w - c)w'\varphi_{1,2}}{T - M^2(w - c)^2} \tag{99}$$

holds (equations (49) to (53)). At the wall, $w = 0$, and

$$f_{1,2} = i\frac{T_1\varphi'_{1,2} + M^2w'_1c\ \varphi_{1,2}}{T_1 - M^2c^2} \tag{99a}$$

Thus, the condition that f vanishes at the wall may be replaced by the condition that a linear combination of $\varphi'_{1,2}, \varphi'_{3,4}$, and $\varphi_{1,2}$ vanishes.

Analogous to the incompressible case (cf. equation (6.8) of reference 1), the condition of boundedness at infinity rules out the solutions f_4 and φ_4. It is convenient to take the lower limit of integration in f_3 and φ_3 at $+\infty$. Then, these solutions vanish rapidly as $y \longrightarrow \infty$. Thus, for $y \gg 1$, the inviscid solutions dominate. In the incompressible case, the inviscid solutions behave like $e^{\pm\alpha y}$. The condition of boundedness therefore leads to $\varphi \sim e^{-\alpha y}$. This is conveniently expressible as

$$\frac{\varphi'}{\varphi} + \alpha \longrightarrow 0 \quad \text{as} \quad y \longrightarrow \infty \tag{100}$$

In the present case, a corresponding condition must be established. However, the situation turns out to be more complicated. Consider the equation

$$\frac{d}{dy}\left\{\frac{(w - c)\varphi' - w'\varphi}{T - M^2(w - c)^2}\right\} - \frac{\alpha^2(w - c)}{T}\varphi = 0 \tag{101}$$

obtained from (54) by dropping the term in $\frac{1}{F^2}$. As will be discussed more in detail in part II, the behavior of the solutions as $y \longrightarrow \infty$ is given by

$$\varphi \sim e^{\pm\beta y}, \quad \text{if} \quad \beta = \alpha\sqrt{\Omega} = \alpha\sqrt{1 - M^2(1 - c)^2} \neq 0 \tag{102}$$

where β is uniquely determined if a cut is drawn along the negative real axis of the complex Ω-plane. Then it is clear that the real part of β is always positive if Ω is not on the cut, and hence the solution $\varphi \sim e^{+\beta y}$ must be rejected. Thus, there results the condition

$$\frac{\varphi'}{\varphi} + \beta \longrightarrow 0 \quad \text{as} \quad y \longrightarrow \infty \tag{103}$$

This reduces to the incompressible case if $M = 0$. If Ω lies on the cut $\left(c = c_r < 1 - \frac{1}{M}\right)$, the situation is more complicated. The situation will become clearer only after a thorough investigation of the inviscid problem (sec. 7).

Except in the case $1 - M^2(1 - c)^2 \leq 0$, the characteristic-value problem is therefore almost the same as that in the incompressible fluid. The characteristic values are given by the determinantal relation

$$\begin{vmatrix} \varphi_{11} & \varphi_{21} & \varphi_{31} \\ \dfrac{T_1\varphi_{11}' + M^2w_1' c\varphi_{11}}{T_1 - M^2c^2} & \dfrac{T_1\varphi_{21}' + M^2w_1' c\varphi_{21}}{T_1 - M^2c^2} & \varphi_{31}' \\ \varphi_{12}' + \beta\varphi_{12} & \varphi_{22}' + \beta\varphi_{22} & 0 \end{vmatrix} = 0$$

where φ_1 and φ_2 are any two linear independent solutions of (101), and

$$\varphi_{ij} = \varphi_i(y_j), \quad \varphi_{ij}' = \varphi_i'(y_j), \quad i,j = 1,2 \tag{104}$$

y_2 being the coordinate of the "edge" of the boundary layer. Strictly speaking, the value $+\infty$ should be substituted for y_2. However, for $y > y_2$, the solution of (101) is practically identical with $e^{\pm\beta y}$. Thus, it is a good approximation to impose equation (103) for $y = y_2$. Naturally, the larger the thickness of the boundary layer is taken, the better is the approximation.

The determinantal equation may be written in the form

$$E(\alpha,c,M^2) = F(z) \tag{105}$$

where $F(z)$ is the function of Tietjens (reference 11)

$$F(z) = 1 + \int_{+\infty}^{-z} H_{1/3}^{(1)}\left[\frac{2}{3}(i\zeta)^{\frac{3}{2}}\right] \zeta^{\frac{3}{2}} \bigg/ z \int_{+\infty}^{z} H_{1/3}^{(1)}\left[\frac{2}{3}(i\zeta)^{\frac{3}{2}}\right]\zeta^{\frac{1}{2}}\, d\zeta \qquad (106)$$

with

$$z = -\left(\frac{w_c'}{\nu_{1c}}\right)^{1/3} \eta_1 \qquad (107)$$

η_1 being the value of η at $y = y_1$, the solid boundary. The function $E(\alpha,c,M^2)$ depends only on the inviscid solutions, and is given by

$$(y_1 - y_c)E(\alpha,c,M^2) = \begin{vmatrix} \varphi_{11} & \varphi_{12}' + \beta\varphi_{12} \\ \varphi_{21} & \varphi_{22}' + \beta\varphi_{22} \end{vmatrix} \div \begin{vmatrix} \dfrac{T_1\varphi_{11}' + M^2 w_1' c\varphi_{11}}{T_1 - M^2c^2} & \varphi_{12}' + \beta\varphi_{12} \\ \dfrac{T_1\varphi_{21}' + M^2 w_1' c\varphi_{21}}{T_1 - M^2c^2} & \varphi_{22}' + \beta\varphi_{22} \end{vmatrix} \qquad (108)$$

The manner in which the viscosity coefficient enters the final equation (106) is noteworthy. As compared with the incompressible case, it amounts only to a change of the definition of z. By referring to (106), (64), and (65), it is seen that this amounts to the replacement of R by R/ν_{1c}. This means that the Reynolds number defined in terms of the free stream velocity and the kinematic viscosity coefficient at the critical layer (instead of that in the free stream) is the quantity governing stability phenomena. This point must be kept in mind whenever it is necessary to compare a case of homogeneous temperature with a case of inhomogeneous temperature. Greater detail will be given in discussing the stability problem in a real fluid (pt. III).

In the case $1 - M^2(1 - c)^2 < 0$, it is possible, of course, also to impose the boundary condition (103), with β imaginary. The same equation (108) holds. But the general discussion of the physical significance of the solutions is more complicated. It will be discussed more in detail in part II.

The inviscid case. In the limit of infinite Reynolds number, the the relation (106) becomes

$$E(\alpha,c,M^2) = 0, \quad \text{or} \quad \begin{vmatrix} \varphi_{11} & \varphi_{12}' + \beta\varphi_{12} \\ \varphi_{21} & \varphi_{22}' + \beta\varphi_{22} \end{vmatrix} = 0 \qquad (109)$$

This corresponds to a solution of (101) with the boundary conditions

$$\varphi(y_1) = 0, \quad \varphi'(y_2) + \beta\varphi(y_2) = 0 \tag{110}$$

Consideration of this boundary problem gives the asymptotic behavior of the relation (61) in the form

$$c = c(\alpha, M^2) \tag{111}$$

This will be discussed fully in the next part.

II - STABILITY IN AN INVISCID FLUID

5. General Considerations

It has been shown that, in the limit of infinite Reynolds numbers, the problem can be treated with viscosity neglected, provided proper care be given to the inviscid solutions. Such investigations will naturally give some information to the stability problem in a viscous and conductive fluid. Indeed, the complete calculation of characteristic values, in particular, of the limit of stability, can be carried out, once the inviscid solutions are known. It is therefore advantageous to study the inviscid case as a prelude to the actual case, with the expectation that certain important characteristics may be obtained.

However, it must be noted that the results obtained in this case cannot be applied directly to the viscous case without modification. Thus, if only stable (damped and neutral) disturbances can exist for a given flow in an inviscid fluid, it cannot be concluded that unstable disturbances cannot exist under the action of viscosity. However, if unstable disturbances exist in the inviscid case, the flow will still be unstable when viscosity is taken into account. For if the continuous dependence of c_i on R is considered, it is evident that c_i cannot remain less than or equal to zero for all <u>finite</u> values of R and still become positive as R becomes infinite.

This investigation will begin with a careful study of the analytical nature of the solutions, especially for y becoming infinite. It is found that the disturbance there takes the form of progressive waves outside the boundary layer. For more detailed discussions of their properties, it is found convenient to classify the disturbances as "subsonic," "sonic," or "supersonic" when the x-component of the phase velocity of the disturbance relative to the free-stream velocity is less than, equal to, or greater than the mean speed of sound in the free stream.

The amplitudes of these waves go to zero as an exponential function of the distance from the solid boundary, except in the case of neutral supersonic disturbances. To an observer moving with the velocity of the free stream, the waves are propagating opposite to his direction of motion for neutral subsonic disturbances. For a general disturbance, the direction of propagation is inclined outward if the wave is amplified and inward if it is damped. For the neutral supersonic disturbances, there may exist both an incident wave and a reflected wave with (in general) non-vanishing amplitudes at infinity.

Analogous to the incompressible case, an attempt is made to establish necessary and sufficient conditions for the existence of certain types of disturbance. But a consideration of energy relations is found to be extremely helpful. This is carried out in section 8.

In the case of neutral or slightly non-neutral subsonic disturbances, the physical situation for the compressible fluid must be quite similar to the situation in the limiting case of an incompressible fluid. Therefore, it should be possible to obtain a general criterion for the existence of slightly amplified subsonic inviscid disturbances, analogous to the Rayleigh-Tollmien criterion for an incompressible fluid (sec. 9b). After such a criterion is developed, mean velocity-temperature profiles could be readily classified according to their relative stability at very large Reynolds numbers, and the effects of the compressibility and conductivity of a gas on the stability of laminar boundary layer flow can be evaluated (sec. 11).

In the case of neutral supersonic disturbances, both incoming and outgoing waves may exist, with the amplitudes of the incident and reflected waves unequal in general. Except in the particular case of a pure neutral outgoing or incoming wave, there is therefore no characteristic-value problem; or rather, the characteristic values are continuous, and not discrete. By utilizing the results of the investigation of the energy balance for a neutral inviscid disturbance (sec. 8) a general expression will be obtained for the ratio of the energy carried out of the boundary layer by the reflected wave to the energy brought into the boundary layer by the incident wave (sec. 10). With the aid of this expression for the "reflectivity," at least a necessary condition for the existence of a pure neutral outgoing or incoming wave can be determined (sec. 10).

6. The Equation of Inviscid Disturbance and the Analytical Nature of the Inviscid Solutions

In the limiting case of infinite Reynolds number and infinite Froude number, the disturbance equation for φ reduces to the following linear differential equation of the second order (cf. (101)):

$$\frac{d}{dy}\left\{\frac{(w-c)\varphi'-w'\varphi}{T-M^2(w-c)^2}\right\} = \alpha^2\frac{(w-c)}{T}\varphi \tag{112}$$

or, in the self-adjoint form,

$$\frac{d}{dy}\left(\xi\frac{d\varphi}{dy}\right) - \left(q + \frac{\alpha^2}{T}\right)\varphi = 0 \tag{113}$$

where

$$\xi(y) = \left\{T-M^2(w-c)^2\right\}^{-1} \qquad q(y) = \frac{1}{w-c}\,\frac{d}{dy}\left(\xi\frac{dw}{dy}\right) \tag{114}$$

There is also the relation

$$\frac{\pi}{p} = -i\gamma M^2 \left\{\frac{(w-c)\varphi' - w'\varphi}{T - M^2(w-c)^2}\right\} = i\gamma\,\frac{\varphi' + if}{w-c} \tag{115}$$

The second part of the last equation may also be written as

$$f = i\,\frac{T\varphi' - M^2 w'(w-c)\varphi}{T - M^2(w-c)^2} \tag{116}$$

Since the coefficients of the differential equation (112) are entire functions of the parameter α^2, the two particular integrals φ_1 and φ_2 of this equation must also be entire functions of α^2. Series developments of φ_1 and φ_2 in powers of α^2 are therefore uniformly convergent for any finite region of α, for a fixed value of y, except when y is a singular point of the differential equation.[1]

If the series development $\varphi = \varphi^{(0)} + \alpha^2\varphi^{(1)} + \alpha^4\varphi^{(2)} + \ldots$ is substituted into (112), two particular integrals φ_1 and φ_2 are obtained by successive quadratures.

$$\varphi_1\,(y;\ \alpha^2,\ c,\ M^2) = (w-c)\sum_{n=0}^{\infty} \alpha^{2n} h_{2n}(y,\ c,\ M^2) \tag{117}$$

$$\varphi_2\,(y;\ \alpha^2,\ c,\ M^2) = (w-c)\sum_{n=0}^{\infty} \alpha^{2n} k_{2n+1}\,(y,\ c,\ M^2) \tag{118}$$

where

$$\left.\begin{aligned} h_{2n}(y;\ c,\ M^2) &= \int_{y_1}^{y}\left\{\frac{T}{(w-c)^2} - M^2\right\}dy \int_{y_1}^{y}\frac{(w-c)^2}{T} \\ &\qquad \times h_{2n-2}(y;\ c,\ M^2)\,dy \\ h_0\,(y;\ c,\ M^2) &= 1 \end{aligned}\right\} \tag{119}$$

[1]It will be shown later (sec. 7b) that the point at which T equals $M^2(w-c)^2$ is only an apparent singularity of (112). The point $y = y_c$ and the point at infinity are the only true singularities.

and

$$\left.\begin{aligned} k_{2n+1}(y;c,M^2) &= \int_{y_1}^{y} \left\{ \frac{T}{(w-c)^2} - M^2 \right\} dy \int_{y_1}^{y} \frac{(w-c)^2}{T} \\ &\qquad \times k_{2n-1}(y;c,M^2)dy, \quad n \geq 1 \\ k_1\ (y;\ c,\ M^2) &= \int_{y_1}^{y} \left\{ \frac{T}{(w-c)^2} - M^2 \right\} dy \end{aligned}\right\} \quad (120)$$

In these integrals, the lower limit is taken at the wall $(y = y_1)$ merely for convenience.

In order that φ_1 and φ_2 may be valid approximations to the regular solutions of the complete disturbance equations (19) to (23) all along the path of integration between the points $y = y_1$ and $y = y$ on the real axis, that path must lie wholly in a region in which the asymptotic expansions of the regular solutions for large values of αR are valid. The asymptotic expansions of these solutions for large values of αR hold only in the range defined by $-\frac{7\pi}{6} < \arg\left\{\left(\frac{w_c'}{\upsilon_{1c}}\right)^{1/3} (y-y_c)\right\} < \frac{\pi}{6}$

and $\quad -\frac{7\pi}{6} < \arg\left\{\left(\frac{\sigma_c w_c'}{\upsilon_{1c}}\right)^{1/3} (y-y_c)\right\} < \frac{\pi}{6} \qquad$ (cf. (95)).

Consequently, the path of integration between y_1 and y must be taken <u>below</u>[1] the point $y = y_c$. (See fig. 1.)

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[1]Contrary to the statement made by Tollmien (reference 4), the proper path must be taken <u>below</u> the point $y = y_c$ regardless of whether this point lies above, on, or below the real axis. This question, which has been responsible for a certain amount of confusion, was finally clarified recently by C. C. Lin (reference 1).

It is now possible to define a region in the complex y-plane in which the solutions φ_1 and φ_2 are everywhere analytic in the variable y and the parameters α^2, c, and M^2. Consider the simply connected region R' which includes the end-points $y = y_1$ and $y = y_2$ but not the (singular) point $y = y_c$ (fig. 2). The region R' and the region S' in the neighborhood of the point $y = y_c$ can be made mutually exclusive. Provided $w'(y) \neq 0$ in the range $y_1 < y < y_2$, the relation $c = w(y)$ maps the regions R' and S' in the y-plane into the mutually exclusive regions R'' and S'' in the complex c-plane.

If y is now restricted to R' and c to R'', the coefficients of (112) are analytic functions of the variable y and the parameters α^2, c, and M^2, and the solutions φ_1 and φ_2 must also be analytic functions of (y_1, α^2, c, M^2). So far as the characteristic-value problem is concerned, the 'analyticity' of the solutions φ_1 and φ_2 in a simply connected region enclosing the boundary points y_1 and y_2 is assured. Unfortunately, this argument fails when $c = 0$, because the singular point $y = y_c$, $w = 0$ coincides with the point $y = y_1$ at the solid boundary, and the regions R' and S' cannot possibly be mutually exclusive. This special case will be discussed briefly in sections 9a and 9b.

7. Further Discussions of the Analytical Nature of the Solutions; Their Behavior around the Singular Points of the Differential Equation

Although the analytical character of the solutions φ_1 and φ_2 in the region R is of great importance for the characteristic-value problem, the behavior of φ_1 and φ_2 in the neighborhood of the singularities of (112) is equally important in the investigation of the physical mechanism of instability.

(a) Singularity at the Point $w = c$:

Unless the quantity $\left[\dfrac{d}{dy}\,(\xi w')\right]_{w=c} = \left[\dfrac{d}{dy}\left(\dfrac{w'}{T}\right)\right]_{w=c}$ vanishes,

the point $y = y_c$ in the complex y-plane, is a regular singularity of the differential equation (112). Since $(w-c)$ and T are analytic functions of y everywhere in the finite region of the complex y-plane, they can be developed in Taylor's series around the point $y = y_c$ $(w = c)$, as follows:

$$w-c = w_c'\,(y-y_c) + \frac{w_c''}{2!}\,(y-y_c)^2 + \ldots \tag{121}$$

$$T = T_c + T_c'\,(y-y_c) + \frac{T_c''}{2!}\,(y-y_c)^2 + \ldots \tag{122}$$

Upon substituting the series developments (121) and (122) into (112), two linearly independent solutions φ_1 and φ_2 valid in the vicinity of the singular point $y = y_c$ are obtained:

$$\varphi_1 = (y-y_c)\; g(y-y_c) \tag{123}$$

$$\varphi_2 = g_2(y-y_c) + \frac{{T_c}^2}{(w_c')^3}\left[\frac{d}{dy}\left(\frac{w'}{T}\right)\right]_{w=c} \varphi_1 \log\,(y-y_c) \tag{124}$$

where g_1 and g_2 are analytic functions of $(y - y_c)$, α^2, c and M^2, and $g_1\ (0) = w_c' \neq 0$, $g_2\ (0) = T_c/w_c' \neq 0$. These two solutions must be subjected to the same restrictions (95) as the solutions (117) and (118). Consequently, in passing from $Rl\ (y - y_c) > 0$ to $Rl\ (y - y_c) < 0$, the correct path lies below the point $y = y_c$, and the proper analytical continuation of (124) for $(y - y_c) < 0$ is [1]

$$\varphi_2 = g_2\,(y-y_c) + \frac{{T_c}^2}{{w'_c}^3}\left\{\frac{d}{dy}\left(\frac{w'}{T}\right)\right\}_c \varphi_1 \left\{\log\left|\,y-y_c\right| - i\pi\right\} \tag{124a}$$

For the physical problem, of course, only the properties of the solutions φ_1 and φ_2 along the real axis are important. If $c_i > 0$ (amplified disturbance) the point $y = y_c$ lies above the real axis, and the solutions are regular along the real axis. In this case, the effect

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[1] Since f is related to φ by (116), the discontinuity suffered by $Im\left\{\varphi_2'\right\}$ in passing from $(y-y_c) < 0$ to $(y-y_c) > 0$ leads to a phase discontinuity in f, and it is this phase shift which makes possible the transfer of energy from the mean flow to the disturbance, or vice versa (sec. 8).

of viscosity and conductivity on the disturbance is negligible in the interior of the fluid for very large Reynolds numbers. However, if $c_i < 0$, the inviscid solution (124) cannot possibly be valid all along the real axis (fig. 1). If $c_i = 0$ (neutral disturbance), there is a critical layer of fluid at the point $w = c$ in which the velocity varies very rapidly $(f \sim \log|y-y_c|)$, and in which, therefore, the viscous forces must be taken into account even when the Reynolds number becomes indefinitely large. If $c_i < 0$, (damped disturbance), there are four such inner critical layers, because the lines $\arg\left\{\left(\frac{w_c'}{\nu_{1c}}\right)^{1/3}(y-y_c)\right\} = -\frac{7\pi}{6}, \frac{\pi}{6}$ (fig. 1) and the lines $\arg\left\{\left(\frac{w_c'}{\nu_{1c}}\right)^{1/3}(y-y_c)\right\} = -\frac{7\pi}{6}, \frac{\pi}{6}$, which delimit the region of validity of the solutions φ_1 and φ_2, intersect the real axis in four points.

With the aid of the equations of motion and the relations (115) and (116), the physical situation in the neighborhood of the point $w = c$ can be made still clearer. It is not difficult to show that the rate of change of the quantity[1] $\rho^*\zeta^*$ where ζ^* is the vorticity, for any two-dimensional motion in an inviscid, non-conductive compressible fluid is given by the relation:

$$\frac{d}{dt^*}(\rho^*\zeta^*) = 2\zeta^*\frac{d\rho^*}{dt^*} + \frac{1}{\rho^*}\frac{\partial(\rho^*,p^*)}{\partial(x^*, y^*)} \tag{125}$$

In the present case, if $c = c_r$, then from (125) and the equations (49) to (53), there is obtained

$$\frac{d}{dt^*}(\bar{\rho}^*\zeta^*) = 0 \quad \text{at} \quad w = c \tag{126}$$

or

$$\varphi\frac{d}{dy}(\rho w') = 0 \quad \text{at} \quad y = y_c \tag{127}$$

that is, the transport of the quantity $\rho w'$ across the plane $w = c$ must vanish. It will be shown later (sec. 8) that it is impossible for $\varphi(y_c) = \varphi_c$ to vanish if $\varphi(y)$ is a solution of the disturbance equation

[1]The quantity $\rho^*\zeta^*$ is related to the density of angular momentum.

(112) which satisfies the boundary condition $\varphi = 0$ at the wall. If the value of $c = c_r$ is chosen so that $\left[\frac{d}{dy}(\rho w')\right]_{w=c} \neq 0$, then the transport of $\rho w'$ across the plane $w = c$ can be balanced only by the diffusion of $\rho w'$ through the action of viscosity. It can therefore be concluded that a neutral disturbance free from the effects of viscosity in the interior of the fluid can exist only for velocity-temperature profiles for which $\frac{d}{dy}(\rho w') = \frac{d}{dy}\left(\frac{w'}{T}\right) = 0$, at some point.

From the energy equation (52), the relation (116), and (123), (124), (124a), it appears that if $c = c_r$ (neutral disturbance), then θ becomes indefinitely large as $w \rightarrow c$, $y \rightarrow y_c$. Even if the quantity $\left[\frac{d}{dy}\left(\frac{w'}{T}\right)\right]_c$ vanishes, the conductivity of the fluid cannot be neglected in the vicinity of the point $w = c$ unless $T' = 0$, which is not generally the case. However, the mathematical results obtained in part I indicate that the influence of the conductivity on the "viscous" solutions of the velocity components is only secondary for Reynolds numbers of the order of magnitude of those encountered in most aerodynamic problems.

(b) Singularity at the Point $T = M^2 (w - c)^2$:

In the case of the neutral supersonic disturbance, outside the boundary layer, the relative velocity between the mean flow and the x-component of the phase velocity of the disturbance is always greater than the mean sonic velocity. At some point within the boundary layer, the relative velocity must be equal to the local mean sonic velocity a. Since $\left(\frac{\bar{a}^*}{\bar{a}^*_0}\right)^2 = \frac{T}{M^2}$, this point will be reached when $(w-c)^2 = \left(\frac{\bar{a}^*}{\bar{u}_0^*}\right)^2 = \frac{T}{M^2}$. Although $\xi(y) \rightarrow \infty$ as $T \rightarrow M^2 (w-c)^2$, by means of a change in dependent variables it is not difficult to show that this point is only an apparent singularity of the differential equation (112). If the dependent variables are chosen as φ and $\frac{\pi}{p}$, rather than φ and φ', then by utilizing (112) and (116), a new system of linear differential equations of the first order is obtained:

$$\frac{d\varphi}{dy} = \frac{i}{\gamma M^2} \cdot \frac{T - M^2(w - c)^2}{(w - c)} \frac{\pi}{p} + \frac{w'}{w - c} \varphi \tag{128}$$

$$\frac{d\left(\frac{\pi}{p}\right)}{dy} = \qquad - i\gamma M^2\alpha^2 \frac{(w - c)}{T} \varphi \tag{129}$$

The only singularities of equations (128) and (129) occur at the point $w = c$ and the point at infinity. So far as the disturbance is concerned, the physical significance of the point $T = M^2 (w - c)^2$ lies only in the fact that it marks the point of transition between the supersonic and subsonic fields of flow.

For the neutral sonic disturbance $\left(1 - c = \frac{1}{M}\right)$, the point at which $(w - c)^2$ equals $\frac{T}{M^2}$ moves out to infinity; $(T \to 1,\ w \to 1,\ \text{as}\ \ y \to \infty)$.

The physical and mathematical problem is more difficult to investigate in this case, because equation (112) has an essential singularity at infinity and the asymptotic behavior of $w(y)$ and $T(y)$ as $y \to \infty$ is somewhat complicated. In the next section, the behavior of the sonic disturbance as $y \to \infty$ will be discussed in some detail.

(c) Behavior of the Inviscid Disturbance as $y \to \infty$

Boundary Conditions and the Characteristic-Value Problem:

As $y \to \infty$, $T \to 1$, $w \to 1$, $w' \to 0$. If $\alpha^2 \left\{1 - M^2(1-c)^2\right\} \neq 0$ the disturbance equation (112) takes the limiting form

$$\varphi'' = \alpha^2 \left\{1 - M^2 (1 - c)^2\right\} \varphi \tag{130}$$

Equation (130) has the solutions $e^{-\beta y}$ and $e^{+\beta y}$, where $\beta = \alpha \sqrt{\Omega}$ and $\Omega = 1 - M^2(1 - c)^2$. It follows that the equation (112) has a fundamental system of solutions behaving like $e^{\pm\beta y}$ as $y \to \infty$. To define β uniquely, it is necessary to introduce a "cut" along the negative real axis of the complex Ω-plane. Regardless of whether $\left\{1 - M^2(1-c_r)^2\right\} \gtreqless 0$, (subsonic, sonic, or supersonic disturbance)[1] the real part of β will be positive so long as $-\pi < \arg(\Omega) < \pi$. Since the physical conditions of the problem require that φ must be bounded as $y \rightarrow \infty$, the solution $e^{+\beta y}$ must be rejected. Therefore, this solution φ must behave like $e^{-\beta y}$ as $y \rightarrow \infty$.

Solutions of the type $e^{-\beta y}$, when combined with the factor $e^{i\alpha(x-ct)}$ evidently represent progressive waves, but it is necessary to be careful in discussing its direction of propagation. A disturbance which is propagated outward with respect to a fixed observer at the wall is actually an incident wave relative to an observer moving with the velocity of the mean flow outside the boundary layer, and vice versa. This fact can be readily appreciated by referring to figure 4. The wave fronts moving outward and also downstream with a velocity c_r relative to the wall are overtaken by the observer moving downstream with the velocity 1 relative to the wall. To such a moving observer, these wave fronts appear to be propagating inward. The situation is obviously reversed for the wave fronts moving inward and downstream with respect to the wall. The same conclusion can be reached, of course, by referring to the analytical form of the disturbance. A wave front moving outward with respect to the fixed wall will have the form $e^{i\alpha(x - ct)} e^{+i\omega y}$. However, for the observer moving with the free stream velocity, the x-coordinate is $x' = x - t$, and the wave front has the form $e^{i\alpha(x' + (1 - c) t)} e^{i\omega y}$ in his (x', y) coordinate system. The wave front is obviously moving inward in this system. If $c_i > 0$ (disturbance increasing with time), then $\Omega_i > 0$ and $\beta_i > 0$; the disturbance takes the form of an outgoing wave of exponentially damped amplitude (in y) as $y \rightarrow \infty$. If $c_i < 0$ (disturbance damped with time), then $\Omega_i < 0$ and $\beta_i < 0$; the disturbance takes the form of an incoming wave of exponentially damped amplitude (in y) as $y \rightarrow \infty$. If $c_i = 0$ and $\Omega > 0$ (neutral subsonic disturbance), the disturbance is propagated parallel to the x-axis, and the amplitude is exponentially damped in y as $y \rightarrow \infty$. Thus, for $-\pi < \arg(\Omega) < \pi$, the boundary condition at $y = y_2$ is $\varphi'(y) + \beta\varphi(y) = 0$, and the characteristic values are discrete (sec. 4).

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[1]The curve $\Omega_i^2 = 4\Omega_r$, corresponding to the condition $1-M^2(1-c_r)^2 = 0$ for a sonic disturbance divides the complex Ω-plane into a region of subsonic disturbances and a region of supersonic disturbances (fig. 3).

If $c_i = 0$ and $\Omega < 0$ (neutral supersonic disturbance), then $\sqrt{\Omega}$ is purely imaginary, and both solutions of equation (130) $\left(e^{\pm\alpha\sqrt{|\Omega|}\,y\,i}\right)$ are bounded as $y \rightarrow \infty$. The corresponding pressure disturbances are also finite (equation (115)). In this case, both incoming and outgoing waves exist, but in general they are not of equal amplitude. This phenomenon can be described physically as a reflection of an incident wave, either with absorption or reinforcement, and will be discussed in more detail in section 10a. Mathematically speaking, in this case there is no homogeneous boundary condition of the type (58) at $y = y_2$. Except for the special case of a pure incoming or a pure outgoing wave, there is therefore no characteristic-value problem, or rather the characteristic values are continuous and not discrete. A solution satisfying the boundary condition at the wall can always be found for arbitrary values of c and α. In fact, from (117) and (118), such a solution for $y_1 < y < y_2$ is

$$\varphi(y) = (w - c) \sum_{n=0}^{\infty} \alpha^{2n} k_{2n+1}(y;\ c, -M^2) \tag{131}$$

The condition $\alpha^2 \left\{1 - M^2(1 - c)^2\right\} \neq 0$ breaks down when $1-M^2(1-c)^2 = 0$, (neutral sonic disturbance) or when $\alpha = 0$. In the latter case, with $1 - M^2(1 - c)^2 \neq 0$, it is not difficult to verify that the solutions (117) and (118) are continuous in α as $\alpha \rightarrow 0$ even as $y \rightarrow \infty$, although the point at infinity of the y-plane is an irregular singularity of the equation (112). Indeed, these solutions behave like $e^{-\beta y}(1 - c)$ and $\dfrac{\sqrt{1 - M^2(1 - c)^2}}{\alpha(1 - c)}\left(e^{-\beta y} - e^{+\beta y}\right)$, respectively, as $y \rightarrow \infty$ in the limiting case $\alpha \rightarrow 0$.

For the case of the neutral sonic disturbance, $\left\{1 - M^2(1 - c)^2\right\} = 0$, and the asymptotic behavior of the inviscid solutions as $y \rightarrow \infty$ is quite complicated. The asymptotic behavior of the mean velocity $w(y)$ for the case of the compressible fluid boundary layer is similar to that of the Blasius profile (reference 11, equation 10, p. 228).

$$1 - w \sim \int_y^{\infty} e^{-\lambda z^2} dz, \quad \text{for } y \gg 1, \quad (\lambda = \text{const.}) \tag{132}$$

In the special case in which the Prandtl number is unity and the mean pressure gradient in the direction of the mean flow is zero, the mean temperature $T(y)$ is a unique quadratic function of $w(y)$. Thus, (reference 11)

$$T = T_1 + \left\{ \frac{\gamma - 1}{2} M^2 - (T_1 - 1) \right\} w - \frac{\gamma - 1}{2} M^2 w^2 \tag{133}$$

In that case, since $c = c_0 = 1 - \frac{1}{M}$,

$$T - M^2 (w - c)^2 \sim (1 - w) F \quad \text{for} \quad y >> 1 \tag{134}$$

where F is a positive constant. The differential equation (112) must take the limiting form:

$$\frac{d}{dy} \left\{ \frac{(w - c)^2}{(1 - w)F} \psi' \right\} = \frac{\alpha^2 (w - c)^2}{T} \psi, \qquad \alpha \neq 0 \tag{135}$$

or,

$$\psi'' + \frac{w'}{1 - w} \psi' = \frac{\alpha^2}{T} (1 - w) F \psi \tag{136}$$

where $\psi = \varphi/(w - c)$. If the physical condition that ψ be bounded as $y \rightarrow \infty$ is imposed, then $\lim_{y \rightarrow \infty} (1 - w)\psi = 0$, and equation (136) admits of two possible solutions:

$$\psi \rightarrow \text{constant}, \quad \varphi \sim w - c \rightarrow \text{constant}, \quad \text{as} \quad y \rightarrow \infty \tag{137}$$

or

$$\psi' \sim 1 - w, \quad \varphi(y) \sim \int_y^\infty (1 - w) dy \rightarrow 0 \quad \text{as} \quad y \rightarrow \infty \tag{138}$$

The asymptotic behavior $\varphi \sim (w - c)$ implies that

$$\frac{d}{dy}\left(\frac{\pi}{p}\right) \longrightarrow \alpha^2, \quad \text{or } \frac{\pi}{p} \sim \alpha^2(y + A) \quad \text{as} \quad y \rightarrow \infty, \quad \text{by equation (129)} \tag{139}$$

If $\varphi(y) \sim \int_y^\infty (1 - w)dy$, then

$$\frac{d\frac{\pi}{p}}{dy} \longrightarrow 0, \ \frac{\pi}{p} \longrightarrow \text{constant} \quad \text{as} \quad y \longrightarrow \infty \tag{140}$$

Thus, for the sonic disturbance, if the pressure disturbance is to remain finite at infinity (equation (140)), then φ must approach zero very rapidly as y becomes infinite (equation (138)). The solution (137) must be rejected, and the characteristic-value problem may be expected to have discrete characteristic values. The condition for the existence of a solution in this case will be discussed in section 10c.

However, if only a finite gradient of pressure disturbance is required, but allowing the pressure disturbance itself to become infinite, both solutions (137) and (138) maybe included, and the characteristic values become continuous. The physical significance (if any) of this solution is not clear. But the situation is somewhat analogous to the case of the steady flow of a compressible fluid in the vicinity of the speed of sound, where small local changes in the cross-sectional area bounded by stream lines produce very large local changes in the velocity and pressure.

It should be noted that if $\alpha = 0$ and $c = 1 - \frac{1}{M}$, both solutions (137) and (138) may be included. (See end of sec. 9a.) From the relation (115), it can be seen that the pressure disturbance remains finite as $y \rightarrow \infty$ in this case.

8. Energy Relations for an Inviscid Disturbance

The disturbances have just been classified into (1) self-excited disturbances propagating outward, $(\mathrm{Im}(\Omega) > 0)$, (2) damped disturbances propagating inward $(\mathrm{Im}(\Omega) < 0)$, (3) neutral disturbances propagating parallel to the x-axis $(\Omega > 0)$, and (4) neutral disturbances propagating both inward and outward $(\Omega \leq 0)$. It is very interesting to consider the energy relations in all these cases.[1]

In the first case, there is no doubt that energy must pass from the mean flow into the disturbance, because the amplitude of the disturbance is being increased and energy is being carried to infinity by the wave at the same time. In the second case, the opposite is true. In the third case, there is apparently no transfer of energy between the mean flow and the disturbance. In the fourth case, energy is being carried in and out by the waves; whether energy will pass from the mean flow to the disturbance, or vice versa, depends upon whether the amplitude of the outgoing wave is greater or less than the amplitude of the incoming wave.

For the two cases of neutral disturbances, (3) and (4), it is possible to clarify the physical situation by considering the time average over a period (which is well defined for neutral oscillations).

Since viscosity and conductivity are disregarded, and the neutral disturbance is harmonic both in x^* and t^*, the average time rate of change of the total energy per unit volume over one period and one wavelength must be zero; that is,

$$\overline{\frac{d\epsilon^*}{dt^*}} = \overline{\rho^* \frac{d}{dt^*}\left\{\frac{(u_i^*)^2}{2}\right\}} + \overline{\rho^* \frac{d}{dt^*}(c_p T^*)} = 0 \tag{141}$$

or

$$\overline{\rho^* \frac{d}{dt^*}\left\{\frac{(u_i^*)^2}{2}\right\}} = -\overline{\frac{dp^*}{dt^*}} \tag{142}$$

By neglecting triple and quadruple correlations, and utilizing the dynamic equations like (49) to (53) to carry out certain reductions, the energy balance for the disturbance is obtained in the following form:

$$\bar{\rho}^* \frac{d}{dt}\overline{\left\{\frac{(u_i^*)^2}{2}\right\}} = -\bar{\rho}^* \overline{u_i^* u_j^*} \frac{\partial \bar{u}_i^*}{\partial x_j^*} - \overline{u_i^{*\prime} \frac{\partial p^{*\prime}}{\partial x_i^*}} = 0 \tag{143}$$

[1]These investigations will also form the basis for the discussion of the necessary and sufficient conditions for the existence of a disturbance (secs. 9 and 10).

Now the quantity $\overline{p^{*\prime} \dfrac{\partial u_i{}^{*\prime}}{\partial x_i}}$ is equal to $\dfrac{\alpha}{2}\ Rl \left\{ \tilde{\pi}(\varphi' + i\alpha f) \right\}$, where the symbol $\sim$ denotes the complex conjugate. From the relation (115), there results $\dfrac{\pi}{p} = i\gamma \dfrac{\varphi' + if}{w - c}$, so that the quantity in the bracket of the above expression is purely imaginary when c is real. Hence, $\overline{p^{*\prime} \dfrac{\partial u_i{}^{*\prime}}{\partial x_i}} = 0$. Using this relation, (143) can be reduced to

$$\bar{\rho}^* \frac{d}{dt} \overline{\left\{ \frac{(u_i{}^*)^2}{2} \right\}} = -\bar{\rho}^* \overline{u_i{}^{*\prime}\ u_j{}^{*\prime}} \frac{\partial \bar{u}_i{}^*}{\partial x_j{}^*} - \frac{\partial}{\partial x_i{}^*} \left(\overline{u_i{}^{*\prime}\ p^{*\prime}} \right) = 0 \qquad (144)$$

The relation (144) holds for every point in the fluid. Hence,

$$\iiint \left\{ -\bar{\rho}^* \overline{u_i{}^{*\prime}\ u_j{}^{*\prime}} \frac{\partial \bar{u}_i{}^*}{\partial x_j{}^*} \right\} dV = \iiint \frac{\partial}{\partial x_i{}^*} \left\{ \overline{p^{*\prime}\ u_i{}^{*\prime}} \right\} dV$$

$$= \iint \overline{p^{*\prime}\ u_n{}^{*\prime}}\ dS \qquad (145)$$

where V is a given volume of fluid, S is the bounding surface, and $u_n{}^{*\prime}$ is the component of the velocity perturbations normal to S. Let V be a rectangular parallelepiped of unit dimensions in the x^* and z^* directions, extending from the solid boundary $(y = y_1)$ "to infinity" in the y^* direction. Then with the condition $v^{*\prime} = 0$ for $y = y_1$, (145) becomes

$$\int_{y_1}^{\infty} -\bar{\rho}^* \overline{u^{*\prime}\ v^{*\prime}} \frac{d\bar{u}^*}{dy^*} dy^* = \int_{\bar{u}^*=0}^{\bar{u}^*=\bar{u}_0^*} \tau^* \frac{d\bar{u}^*}{dy^*} dy^* = \lim_{y \to \infty} \overline{p^{*\prime}\ v^{*\prime}} \qquad (146)$$

that is, the net energy propagated outward by the disturbance in unit time across the plane y = constant. (y large) is equal to the total energy transferred in unit time from the mean flow to the disturbance by the action of the shear stress $\tau^* = -\overline{\rho}^* \overline{u^{*\prime}v^{*\prime}}$ within the boundary layer.

It can be verified that

$$\frac{\overline{u^{*\prime}v^{*\prime}}}{\overline{u}_0^{*2}} = \frac{\alpha}{2} Rl\ (f\tilde{\varphi}) \tag{147}$$

where $\tilde{\varphi}$ denotes the complex conjugate of φ. By making use of (116), the expression (147) for the velocity correlation can be brought into the following form when c is real:

$$\frac{\overline{u^{*\prime}v^{*\prime}}}{\overline{u}_0^{*2}} = -\frac{\alpha}{2}\frac{T}{T - M^2\ (w-c)^2}\ \mathrm{Im}\ (\varphi'\tilde{\varphi}) \tag{148}$$

or rather

$$\frac{\overline{u^{*\prime}v^{*\prime}}}{\overline{u}_0^{*2}} = -\frac{\alpha}{2}\frac{T}{T - M^2\ (w-c)^2}\ (\varphi_r\varphi_i' - \varphi_i\varphi_r') \tag{149}$$

When $c = c_r$, the coefficients of the differential equation (112) are real. If φ satisfies (112), then φ_r and φ_i must also satisfy the equation independently, and the expression in brackets in (149) is the Wronskian of the two solutions, by definition. For equation (112), the Wronskian is

$$W\ (\varphi_r,\ \varphi_i) = e^{-\int \frac{\xi'}{\xi} dy} = k\left\{T - M^2\ (w-c)^2\right\} \tag{150}$$

from the relation (114), where k is a real constant. Hence, from (149),

$$\frac{\overline{u^{*\prime}v^{*\prime}}}{(\bar{u}_o^*)^2} = -\frac{\alpha}{2}kT, \quad \tau = \frac{\tau^*}{\bar{\rho}_o^* \, \bar{u}_o^{*\,2}} = \frac{\alpha}{2}k \tag{151}$$

Thus, if $c = c_r$, the shear stress is constant wherever $W(\varphi_r, \varphi_i)$ is continuous, that is, outside of the inner critical layer at $w = c$, where the effects of viscosity and conductivity predominate (sec. 7a). To satisfy the boundary condition at the wall, $\varphi_r(y_1)$ and $\varphi_i(y_1)$ must be zero independently, and therefore, $W = k\left[T_1 - M^2(w_1 - c)^2\right] = 0$ In general, $c^2 \neq T_1/M^2$, so that $k = 0$ and

$$W = 0, \qquad \tau = 0 \quad \text{for} \quad y - y_c < 0 \tag{152}$$

(Cf. fig. 5.)

By utilizing the solutions (123), (124), and (124a), and equation (150) the discontinuity suffered by the Wronskian in passing from $(y < y_c)$ to $(y > y_c)$ can be calculated. In fact[1],

$$\Delta W = \pi \left|\varphi_c\right|^2 \frac{T_c}{w'_c}\left[\frac{d}{dy}\left(\frac{w'}{T}\right)\right]_c \tag{153}$$

where $\varphi_c = \varphi(y_c)$. From (150), $W = kT_c$ for $y = y_c + 0$, and therefore,

$$k = \pi \frac{\left|\varphi_c\right|^2}{w_c'}\left[\frac{d}{dy}\left(\frac{w'}{T}\right)\right]_c \tag{154}$$

and

$$\tau = \frac{\alpha}{2}k \quad \text{for} \quad y - y_c > 0 \tag{155}$$

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[1]In the limiting case of an incompressible fluid, ΔW reduces to the value calculated by Tollmien (reference 10).

Thus, assuming for the moment that $\varphi_c \neq 0$, if the sign of the quantity $\left[\frac{d}{dy}\left(\frac{w'}{T}\right)\right]_c$ is positive, energy will pass from the mean flow to the disturbance; if the sign of $\left[\frac{d}{dy}\left(\frac{w'}{T}\right)\right]_c$ is negative, the mean flow will absorb energy from the disturbance; if $\left[\frac{d}{dy}\left(\frac{w'}{T}\right)\right]_c = 0$, there is no exchange of energy between the mean flow and the disturbance.

In the foregoing discussion, it was tacitly assumed that $\varphi_c \neq 0$. By means of a proof similar to that given by Tollmien (reference 10) for the case of an incompressible fluid, it is not difficult to show that φ_c cannot vanish if $\varphi(y)$ is a solution of (112) satisfying the boundary condition $\varphi(y_1)$ at the wall. This presently will be done.

If $\varphi(y)$ satisfies (112), then $\varphi = A_1 \varphi_1 + A_2 \varphi_2$, and the behavior of φ_1 and φ_2 in the neighborhood of the point $y = y_c$ is given by (123), (124), and (124a). Since $\varphi_1(y_c) = 0$ and $\varphi_2(y_c) \neq 0$ $A_2 = 0$ if $\varphi(y_c)$ vanishes. Now $\varphi_1(y)$ is analytic and $\frac{d}{dy}\left(\frac{\varphi}{w - c}\right)$ is finite at $y = y_c$. By direct integration of equation (112),

$$\varphi(y) = (w-c)\left\{1+\alpha^2 \int_{y_c}^{y} \left\{\frac{T}{(w-c)^2} - M^2\right\} dy \int_{y_c}^{y} \frac{(w-c)^2}{T} dy + \ldots\right\} \tag{156}$$

If $y_1 \leq y \leq y_4$ where y_4 is the value of y for which $\frac{T}{M^2} = (w - c)^2$, then $\left\{\frac{T}{(w - c)^2} - M^2\right\} > 0$, and the quantity in brackets in (156) is positive. Therefore, $\varphi(y) > 0$ for $y_o < y < y_4$ and $\varphi(y) < 0$ for $y_1 < y < y_o$; $\varphi(y)$ can never satisfy the boundary condition $\varphi = 0$ at the wall, and the assumption $\varphi(y_c) = 0$ must be abandoned.

9. Necessary and Sufficient Conditions for the Existence of an Inviscid Subsonic Disturbance

So far in the discussion of the inviscid disturbance in a compressible fluid, it has been assumed that solutions of equation (112) exist which satisfy the given types of boundary conditions. The energy criteria developed in section 8 not only serve to clarify the physical problem considerably, but also lead directly to the formulation at least of the necessary conditions for the existence of each of the three possible types of neutral inviscid disturbance. The sufficiency conditions cannot follow directly from energy considerations.

This section will deal with subsonic disturbances, neutral and self-excited. First a necessary and sufficient condition for the existence of neutral subsonic disturbances is established. It is then possible to establish a sufficient condition for the existence of self-excited subsonic disturbances. However, a necessary condition has not yet been established.

(a) The Neutral Subsonic Disturbance:

At large distances from the wall, the neutral subsonic disturbance dies off like $e^{-\beta y}$, and $\left\{\overline{p^{*\prime}v^{*\prime}}\right\} \longrightarrow 0$ as $y \longrightarrow \infty$. In this case, no energy is transported into or out of the boundary layer by the disturbance, and therefore there is no net exchange of energy between the mean flow and the disturbance within the boundary layer (cf. (146)). From the results of section 8, $\tau \cong 0$, and hence the quantity $\left[\frac{d}{dy}\left(\frac{w'}{T}\right)\right]_c$ must vanish, if $c \neq 0$. If $\frac{d}{dy}\left(\frac{w'}{T}\right)$ does not vanish for some $w > 1 - \frac{1}{M}$, the only possible neutral subsonic disturbance is the one for which $c = 0$. When $\frac{d}{dy}\left(\frac{w'}{T}\right)$ vanishes, for $w = c_s$ (say), then c equals c_s for the neutral subsonic disturbance.

The condition that $\frac{d}{dy}\left(\frac{w'}{T}\right)$ must vanish for some $w > 1 - \frac{1}{M}$ is also sufficient for the existence of a neutral subsonic disturbance. As in the case of an incompressible fluid, the sufficiency condition can be derived by means of an argument based on the fact that $\varphi(y_1; \alpha)$ is an analytic function of α (reference 1). For the purpose of this discussion, it is convenient to deal with the disturbance equation in the self-adjoint form (113). Suppose now that $\frac{d}{dy}\left(\frac{w'}{T}\right) = 0$ for some $y > y_1$, where $w = c_s$ (say). Then, by the necessary condition, the phase velocity of the neutral disturbance, if it exists, must be equal to c_s. Now, $\xi(y)$ is positive continuous and bounded everywhere, and hence $q(y)$ is also continuous and bounded everywhere. Equation (113) can then be integrated directly to give the relation

$$\xi(y)\varphi'(y) = \xi(y_2)\varphi'(y_2) - \int_y^{y_2}\left(q + \frac{\alpha^2}{T}\right)\varphi\, dy \tag{157}$$

By choosing the value of α^2 large enough, the quantity $q + \frac{\alpha^2}{T}$ can always be made positive, since $q(y)$ is bounded. Now, for $c = c_s$, the solution $\varphi(y)$ can be completely defined by the boundary conditions $\varphi(y_2) = 1 - c_s$, $\varphi'(y_2) + \alpha\sqrt{1 - M^2(1 - c_s)^2}\,\varphi(y_2) = 0$. Therefore, $\varphi'(y_2) < 0$ when $\alpha > 0$, and from (157), $\varphi'(y) < 0$ when $q + \frac{\alpha^2}{T} > 0$. Hence, the value of α can be chosen large enough so that $\varphi(y_1) > \varphi(y_2) > 0$. For $\alpha = 0$, however, $\varphi(y) = w - c_s$ and $\varphi(y_1) < 0$. Since $\varphi(y_1; \alpha)$ is a bounded, continuous function of α (sec. 2), $\varphi(y_1)$ must vanish for some value of $\alpha = \alpha_s > 0$. For a given value of the Mach number, the value of $c = c_s$ is determined from the mean velocity-temperature profile, and the corresponding value of the frequency $\alpha = \alpha_s$ is given by the secular equation (109). The boundary-value problem for the case of a neutral subsonic disturbance is completely solved.

From the disturbance equation (112) and the boundary conditions, it can be seen that for $M < 1$ the singular solution $\varphi = w$, for $c = 0$, $\alpha = 0$ (infinite wavelength and zero wave velocity) always exists, provided $\lim_{c,\alpha \to 0} \frac{w_1' c \sqrt{1 - M^2}}{T_1 \alpha} = 1$. If $M > 1$, then in the limiting case of infinite wavelength, $(\alpha = 0)$ the neutral subsonic disturbance becomes a neutral sonic disturbance $\left(c \to 1 - \frac{1}{M}\right)$; that is, the condition

$$\lim_{\substack{c \to \left(1 - \frac{1}{M}\right) \\ \alpha \to 0}} + \frac{w_1' c\, M^2 \sqrt{1 - M^2 (1 - c)^2}}{T_1\, \alpha} = \text{constant}$$

holds. The solution for φ is a linear combination of $(w - c)$ and

$$(w - c) \int_y^{\infty} \left\{ \frac{T}{(w - c)^2} - M^2 \right\} dy.$$

If $M = 1.0$, the condition

$$\lim_{\substack{c \to 0 \\ \alpha \to 0}} + \frac{w_1'\, c^{3/2}}{T_1\, \alpha} = \text{constant}$$

holds for the neutral subsonic disturbance in the limiting case of infinite wavelength, and the singular solution $\varphi = w$ exists. The significance of these limiting conditions will be appreciated in the investigation of the asymptotic behavior of the $\alpha - R$ curve for the neutral subsonic disturbance in a viscous compressible fluid, which will be carried out in a subsequent report.

(b) Amplified and Damped Subsonic Disturbances:

It has been found (sec. 9) that the condition that $\frac{d}{dy}\left(\frac{w'}{T}\right)$ must vanish for some $w > 1 - \frac{1}{M}$ is necessary and sufficient for the existence of a neutral inviscid subsonic disturbance. By analogy with the case of an incompressible fluid, it can be expected that the condition $\frac{d}{dy}\left(\frac{w'}{T}\right) = 0$ for some $w > 1 - \frac{1}{M}$ is also sufficient for the existence of amplified subsonic disturbances ("adjacent" to the neutral subsonic disturbance $c = c_s$, $\alpha = \alpha_s$). If $\frac{d}{dy}\left(\frac{w'}{T}\right)$ does not vanish for some value of $w > 1 - \frac{1}{M}$, it appears probable that except for the disturbances

$c = 0$, $\alpha = 0$, for $M < 1$, or $c = 1 - 1/M$, $\alpha = 0$, for $M \geqq 1$, only damped subsonic disturbances can exist in the inviscid compressible fluid.

To prove that the condition $\frac{d}{dy}\left(\frac{w'}{T}\right) = 0$ for some $w > 1 - \frac{1}{M}$ is a sufficient condition for the existence of amplified disturbances, a method is employed which is quite similar to that used in the incompressible case (reference 1). The following points are settled: (1) The existence of characteristic values of c and α near (c_s, α_s) such that $\operatorname{Im}(c) = c_i \neq 0$; (2) the sign of c_i.

(1) It has already been found (sec. 7) that the boundary conditions for the subsonic disturbance yield a unique relation between the characteristic values of the form

$$c = c\ (\alpha,\ M^2) \tag{158}$$

where c is an analytic function of α and M^2 except in the neighborhood of the point $c = 0$, $\alpha = 0$. In the neighborhood of $\alpha = \alpha_s \neq 0$, there is at least one value of c for every value of α (real), and if $\alpha \neq \alpha_s$ then $c \neq c_s$, and c must be complex; that is, $c_i \neq 0$, for the only permissible real value of c is c_s.

(2) In the neighborhood of $\alpha = \alpha_s > 0$, α and α^2 are uniquely related, and c is an analytic function of α^2. Then $c(\alpha^2)$ may be expanded in a Taylor's series of $\lambda - \lambda_s = \alpha^2 - \alpha_s^2$ around the point $\lambda_s = \alpha_s^2$:

$$c = c_s + (\lambda - \lambda_s)\left(\frac{dc}{d\lambda}\right)_s + \frac{(\lambda - \lambda_s)^2}{2!}\left(\frac{d^2c}{d\lambda^2}\right)_s + \ldots \tag{159}$$

If $\operatorname{Im}\left(\frac{dc}{d\lambda}\right)_s \neq 0$, or, if $\operatorname{Im}\left(\frac{d^kc}{d\lambda^k}\right)_s = 0$, $k = 1, 2, 3, n-1$, but $\operatorname{Im}\left(\frac{d^nc}{d\lambda^n}\right) \neq 0$, and n is odd, then c_i will always be positive for some value of α slightly smaller or larger than α_s. For these values of (c, α), a solution $\varphi(y)$ exists which is valid all along the real axis (sec. 7a).

The imaginary part of $\left(\frac{d^k c}{d\lambda^k}\right)_s$ can most readily be calculated by successive differentiation of the differential equation (112) with respect to λ. If $\varphi(y)$ is a characteristic function, and c and λ are the corresponding characteristic values, $\frac{d\varphi}{d\lambda}$ exists in the regions R' and R'' (sec. 6) and indeed

$$\frac{d\varphi}{d\lambda} = \varphi_\lambda = \frac{\partial\varphi}{\partial\lambda} + \frac{\partial\varphi}{\partial c}\frac{dc}{d\lambda} \tag{160}$$

For the purpose of this discussion, equation (112) can be rewritten in the form

$$L(\varphi) = \varphi'' + \frac{\xi'}{\xi}\varphi' - \left(q + \frac{\lambda}{T}\right)\frac{\varphi}{\xi} = 0 \tag{161}$$

where the primes denote differentiation with respect to y. By differentiating (161) once with respect to λ, the following differential equation for φ_λ is obtained:

$$\begin{aligned} L(\varphi_\lambda) &= \varphi_\lambda'' + \frac{\xi'}{\xi}\varphi_\lambda' - \left(q + \frac{\lambda}{T}\right)\varphi_\lambda/\xi \\ &= \left\{-\varphi'\frac{d}{dc}\left(\frac{\xi'}{\xi}\right) + \left(q + \frac{\lambda}{T}\right)\varphi\frac{d}{dc}\left(\frac{1}{\xi}\right) + \frac{\varphi}{\xi}\frac{dq}{dc}\right\}\left(\frac{dc}{d\lambda}\right) + \frac{\varphi}{T}\frac{1}{\xi} \end{aligned} \tag{162}$$

When $\lambda \rightarrow \lambda_s$, $c \rightarrow c_s$, the corresponding expressions for $L(\varphi_\lambda)$ and $L(\varphi)$ will be denoted by $L_s(\varphi_{\lambda s})$ and $L_s(\varphi_s)$, respectively. From (160) and (161),

$$\xi_s \left\{ \varphi_s L_s(\varphi_{\lambda s}) - \varphi_{\lambda s} L_s(\varphi_s) \right\} = \frac{d}{dy} \left\{ \xi_s \ (\varphi_s \varphi_{\lambda s}' - \varphi'_s \varphi_{\lambda s}) \right\}$$

$$= \frac{\varphi_s^2}{T} + \left(\frac{dc}{d\lambda}\right)_s \left\{ \varphi_s^2 \left(\frac{dq}{dc}\right)_s + \left(q_s + \frac{\lambda_s}{T}\right) \xi_s \varphi_s^2 \left[\frac{d}{dc}\left(\frac{1}{\xi}\right)\right]_s - \varphi_s \varphi'_s \xi_s \left[\frac{d}{dc}\left(\frac{\xi'}{\xi}\right)\right]_s \right\} \tag{163}$$

If both sides of equation (163) are integrated between the limits $y = y_1$ and $y = y_2$ along any path in the region R in the complex y-plane, an expression for $\left(\frac{dc}{d\lambda}\right)_s$ is obtained. Consider first the integral of the left-hand side of equation (163):

$$\int_{y_1}^{y_2} \frac{d}{dy} \left[\xi_s \ \left(\varphi_s \varphi'_{\lambda s} - \varphi'_s \varphi_{\lambda s}\right) \right] dy = \xi_s \left(\varphi_s \varphi'_{\lambda s} - \varphi'_s \varphi_{\lambda s}\right) \Big|_{y_1}^{y_2} \tag{164}$$

Since $\varphi \ (y_1; \ c, \ \lambda) = 0$ is an identity in λ, $\varphi_s(y_1) = \varphi_{\lambda s}(y_1) = 0$, and the integrated expression vanishes at y_1. At the upper limit,

$$\varphi'_s \ (y_2, \ c, \ \lambda) = - \alpha_s \sqrt{1 - M^2 \ (1 - c_s)^2} \ \varphi_s(y_2; \ c_s, \ \lambda_s),$$

$$\varphi'_{\lambda s} \ (y_2, \ c_s, \ \lambda_s) = - \alpha_s \sqrt{1 - M^2 \ (1 - c_s)^2} \ \varphi_{\lambda s} \ (y_2; \ c_s, \ \lambda_s)$$

$$- \varphi_s \ (y_2; \ c_s, \ \lambda_s) \left\{ \frac{d}{d\lambda} \alpha \sqrt{1 - M^2 \ (1 - c)^2} \right\}_s$$

Substituting these relations into (164) finally results in

$$\xi_s\left(\varphi_s\varphi'_{\lambda s} - \varphi'_s\,\varphi_{\lambda s}\right)\Big|_{y_1}^{y_2} = a_1 + a_2\left(\frac{dc}{d\lambda}\right)_s \tag{165}$$

where a_1 and a_2 are real constants.

The integral of the right-hand side of equation (164) is

$$\int_{y_1}^{y_2}\frac{{\varphi_s}^2}{T}\,dy + \left(\frac{dc}{d\lambda}\right)_s\left\{\int_{y_1}^{y_2}\left[{\varphi_s}^2\left(\frac{dq}{dc}\right)_s + \left(q_s + \frac{\lambda_s}{T}\right)\xi_s{\varphi_s}^2\left(\frac{d}{dc}\,\frac{1}{\xi}\right)_s\right.\right.$$

$$\left.\left. - \varphi_s\varphi_s'\xi_s\left(\frac{d}{dc}\,\frac{\xi'}{\xi}\right)_s\right]dy = \int_{y_1}^{y_2}\frac{{\varphi_s}^2}{T}\,dy + \left(\frac{dc}{d\lambda}\right)_s I_1 \tag{166}$$

Equations (164), (165), and (166) yield

$$\left(\frac{dc}{d\lambda}\right)_s = \frac{-\displaystyle\int_{y_1}^{y_2}\frac{{\varphi_s}^2}{T}\,dy + a_1}{I_1 - a_2} \tag{167}$$

In evaluating the integrals I_1 and $\int_{y_1}^{y_2}\frac{{\varphi_s}^2}{T}\,dy$, integrate along the real axis, except for the term $\int_{y_1}^{y_2}{\varphi_s}^2\left(\frac{dq}{dc}\right)_s dy$. Indeed, all the other integrals have real and finite integrands along the real axis. Thus, the imaginary term in (167) can occur only with the integral

$\int_{y_1}^{y_2} \varphi_s^{\,2} \left(\frac{dq}{dc}\right)_s \, dy$, the integrand of which becomes infinite at $y = y_c$.

By expanding the integrand in power series in the neighborhood of $y = y_c$, there is obtained

$$\operatorname{Im}\left\{I_1\right\} =$$

$$= \operatorname{Im}\left\{\int_{y_1}^{y_2} \frac{\left\{\varphi_{sc}^2 + 2\varphi_{sc}\varphi'_{sc}(y-y_c) + \ldots\right\}\left\{(\xi_s w')_c''\,(y-y_c) + \ldots\right\}}{(w'_c)^2\,(y-y_c)^2\left\{1 + \frac{w''_c}{2w'_c}(y-y_c) + \ldots\right\}^2}\,dy\right\} \tag{168}$$

from which,

$$\operatorname{Im}\left\{I_1\right\} = \pi\,\left|\varphi_{sc}\right|^2 \frac{(\xi_s w')''_c}{(w_c')^2} \tag{169}$$

If $\varphi_s(y)$ is a characteristic function, φ_{sc} can never vanish (sec. 8).

Hence, $\operatorname{Im}\left\{I_1\right\} \neq 0$, $\operatorname{Im}\left(\frac{dc}{d\lambda}\right)_s \neq 0$, provided $(\xi_s w')''_c \neq 0$; and

c_i <u>must be positive for some value of α slightly smaller or larger than α_s</u>.

The restriction that $(\xi_s w')''_c$ must not vanish can easily be removed by an extension of the foregoing argument. By the physical nature of the mean velocity-temperature profile, if $(\xi_s w')'_c = 0$ the quantity $\xi_s w'$ must have a true extremum at the point $y = y_c$ $w = c_s$ and not an inflection; in other words, $(\xi_s w')'$ must have a zero of odd order at the point $y = y_c$. Therefore,

$$\left(\xi_s w'\right)' = \frac{(y-y_c)^{2m+1}}{(2m+1)!} \left(\xi_s w'\right)_c^{(2m+2)} + \ldots, \quad m = 0, 1, 2, \ldots \tag{170}$$

From (169), it can be seen that $\operatorname{Im}\left(\frac{dc}{d\lambda}\right)_s = 0$ for $m \geq 1$. The differential equation (162) for φ_λ is regular in the vicinity of the point $y = y_c$, and therefore $\varphi_\lambda(y)$ is real. By differentiating (162) successively with respect to λ, differential equations are obtained for $\frac{d^2\varphi}{d\lambda^2}, \frac{d^3\varphi}{d\lambda^3}, \ldots, \frac{d^k\varphi}{d\lambda^k}$, which are all regular in the vicinity of the point $y = y_c$ if $k < 2m+1$. Consequently, $\varphi_\lambda^{(k)}$ is real for $k < 2m+1$. In the expression for $\left(\frac{d^k c}{d\lambda^k}\right)_s$ a term of the form

$$k!\left(\frac{dc}{d\lambda}\right)^k \int_{y_1}^{y_2} \frac{\varphi_s^2}{(w - c_s)^{k+1}} \left(\xi_s w'\right)' dy$$

always appears by analogy with (168). (All the other terms in the expression for $\left(\frac{d^k c}{d\lambda^k}\right)_s$ are always real.) By virtue of (170), the imaginary part of this term vanishes if $k < 2m + 1$. However, if $k = 2m+1$, the imaginary part of this term is

$$\pi \left(\frac{dc}{d\lambda}\right)^{2m+1} \frac{\left|\varphi_c\right|^2}{(w'_c)^{2m+2}} \left(\xi_s w'\right)_c^{(2m+2)} \neq 0$$

Therefore, $\operatorname{Im}\left(\frac{d^{2m+1} c}{d\lambda^{2m+1}}\right) \neq 0$, and $c_i > 0$ for some value of α slightly larger or smaller than α_s. The proof of the existence of amplified subsonic disturbances adjacent to the neutral subsonic disturbance $c = c_s$, $\alpha = \alpha_s$ is thus complete.

It is quite difficult to give a rigorous proof of the existence of amplified subsonic disturbances adjacent to the neutral disturbance $c = 0$, $\alpha = 0$, chiefly because c is not an analytic function of α in the vicinity of this point. (See secs. 7 and 7c.) Although it does not seem worth while to discuss the details, it can be shown by a method similar to that utilized by Tollmien (reference 10) that $\left(\frac{dc}{d\lambda}\right)_{c=0}$ is real, $\operatorname{Im}\left(\frac{d^2c}{d\lambda^2}\right)_{c=0}$ is positive if $\left[\frac{d}{dy}\left(\frac{w'}{T}\right)\right]_{y_1} > 0$, and $Rl\left(\frac{d^2c}{d\lambda^2}\right)_{c=0}$ is unfortunately logarithmically infinite. The argument in this case is therefore inconclusive. Of course, from the asymptotic behavior of the neutral α-R curve, it should be possible to see that amplified subsonic disturbances do actually exist in the neighborhood of the neutral disturbance $c = 0$, $\alpha = 0$, if the neutral disturbance $c = c_s$, $\alpha = \alpha_s$ exists.

10. Some Further Discussions of Inviscid Disturbances

So far, only subsonic disturbances which are neutral or nearly neutral have been discussed. These disturbances correspond to the <u>immediate neighborhood</u> of the positive real axis of the complex Ω-plane (fig. 3). It has not yet been possible to get any result regarding general <u>non-neutral</u> modes, except that they possess the property of being either self-excited and outgoing, or damped and incoming, and that no sharp change in property would be expected in passing from subsonic to sonic and to supersonic disturbances. The neutral sonic and supersonic disturbances, however, do enjoy a special position. The former corresponds to the branch point of $\sqrt{\Omega}$ at the origin, and the latter corresponds to the cut[1] drawn in the Ω-plane to separate the two solutions $\exp\left(\pm\alpha\sqrt{\Omega}\, y\right)$.

In the following sections, the neutral supersonic disturbance is first considered. The transfer of energy is made the basis of this investigation. In section 10c, the case of the neutral sonic disturbance is discussed briefly.

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[1]Of course any other cut might have been used. This particular one, however, has the desirable property that one of the solutions $\exp\left\{\pm\alpha\sqrt{\Omega}\, y\right\}$ is in general ruled out by physical requirements.

(a) <u>Necessary Conditions for the Existence of a Neutral Supersonic Disturbance:</u>

The results of the investigation of the energy balance for any type of neutral inviscid disturbance lead directly to the <u>necessary</u> conditions for the existence of the neutral supersonic disturbance:

(1) If the mean flow in the boundary layer absorbs energy from the disturbance, $\left[\frac{d}{dy}\left(\frac{w'}{T}\right)\right]_{y=y_c} < 0$, $c > c_s$. The amplitude of the reflected wave must be less than the amplitude of the incident wave.

(2) If there is no exchange of energy between the mean flow and the disturbance, $c = c_s$. The amplitudes of the incident and reflected waves are equal.

(3) If the disturbance absorbs energy from the mean flow in the boundary layer, $\left[\frac{d}{dy}\left(\frac{w'}{T}\right)\right]_{y=y_c} > 0$, $c < c_s$. The amplitude of the reflected wave must be greater than the amplitude of the incident wave. Of course, $c < 1 - \frac{1}{M}$, and α is arbitrary. The stationary Mach waves $c = 0$ (α arbitrary) can always exist.

The necessary conditions for the existence of a pure outgoing or a pure incoming wave will be discussed in connection with the reflection and absorption of the neutral supersonic disturbance. (See sec. 10b.) Formulation of the <u>sufficient</u> conditions in this special case has proved to be a formidable task. In general, $q(y)$ is not bounded at the point $w = c$, and $\xi(y)$ is not bounded at the point $T = M^2 (w - c)^2$. Consequently, it is difficult to determine the sign of $\varphi(y_1; \alpha)$ for large values of α, and it has not yet been possible to carry through the type of argument which served in the case of a subsonic disturbance. (See sec. 9a.)

(b) <u>Reflection and Absorption of the Neutral Supersonic Disturbance:</u>

By the action of the viscous forces within the critical layer at $w = c$ $(c \neq c_s)$, a relative phase shift is produced between $u^{*\prime}$ and

$v^{*\prime}$, and the shear stress $\tau = \frac{\tau^*}{\bar{\rho}_0^*(\bar{u}_0^*)^2}$ increases (or decreases) rapidly from zero for $(y - y_c) < 0$ to the value $\frac{\alpha}{2} k$ for $(y - y_c) > 0$ (secs. 4 and 6). Thus, the critical layer takes the place, in a sense, of a wavy wall or irregular solid boundary, in reinforcing or partially canceling $\left(\text{depending on the sign of } \left[\frac{d}{dy}\left(\frac{w'}{T}\right)\right]_c\right)$ an incoming disturbance during the process of reflection.

From equation (148), the shear stress $\tau = \frac{\alpha}{2} k$ is also equal to the expression $\frac{\alpha}{2} \frac{1}{T - M^2(w - c)^2} \operatorname{Im} (\varphi'\tilde{\varphi})$. Since $\varphi \sim Ae^{i\omega y} + Be^{-i\omega y}$ for $y \gg 1$ (sec. 7)

$$\tau = \frac{\alpha}{2} k = \frac{\alpha^2}{2\sqrt{M^2(1 - c)^2 - 1}} \left\{ |B|^2 - |A|^2 \right\} \tag{171}$$

and hence, from equation (154),

$$\frac{1}{\sqrt{M^2(1 - c)^2 - 1}} \left\{ |B|^2 - |A|^2 \right\} = \pi \frac{|\varphi_c|^2}{w_c'} \left[\frac{d}{dy}\left(\frac{w'}{T}\right)\right]_c \tag{172}$$

By making use of equation (172) and the additional relation

$$|\varphi'(y_2)|^2 + \omega^2 |\varphi(y_2)|^2 = 2\omega^2 \left\{ |A|^2 + |B|^2 \right\} \tag{173}$$

an expression is obtained for the "reflectivity" $K = \frac{|B|^2}{|A|^2}$, defined as the ratio of the energy carried out of the boundary layer by the reflected wave to the energy brought into the boundary layer by the incident wave. It is found that

$$K = \frac{1 + j}{1 - j} \tag{174}$$

where

$$j = \frac{\dfrac{2\pi\omega^3}{\alpha^2}\left[\dfrac{1}{w'}\dfrac{d}{dy}\left(\dfrac{w'}{T}\right)\right]_{y = y_c}}{\left|\dfrac{\varphi'(y_2)}{\varphi_c}\right|^2 + \omega^2\left|\dfrac{\varphi(y_2)}{\varphi_c}\right|^2}$$

It follows that

$$K \gtreqless 1 \quad \text{when} \quad j \gtreqless 0 \quad (\,|j| \leq 1) \tag{175}$$

The necessary condition for the existence of a pure outgoing or incoming wave is

$$\left.\begin{aligned} j &= 1 \\ j &= -1 \end{aligned}\right\} \tag{176}$$

In this case, the boundary condition at $y = y_2$ is

$$\varphi'(y_2) \pm i\omega\varphi(y_2) = 0$$

and

$$\frac{\pi\omega}{\alpha^2}\,\frac{\left[\dfrac{1}{w'}\dfrac{d}{dy}\left(\dfrac{w'}{T}\right)\right]_{y_c}}{\left|\dfrac{\varphi(y_2)}{\varphi_c}\right|^2} = \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} \tag{177}$$

In section 7c, it was remarked that a solution of the disturbance equation (112) satisfying the boundary condition $\varphi = 0$ at the wall ($y = y_1$, or $y = 0$) could always be found for arbitrary values of c and α in the case of the supersonic disturbance. Such a solution is (cf. (131)

$$\varphi(y) = (w - c) \sum_{n=0}^{\infty} \alpha^{2n} k_{2n+1} (y; c, M^2) \tag{178}$$

so that

$$\varphi(y_2) = (1 - c) \sum_{n=0}^{\infty} \alpha^{2n} k_{2n+1} (y_2; c, M^2) \tag{179}$$

$$\varphi'(y_2) = (1 - c) \sum_{n=0}^{\infty} \alpha^{2n} k'_{2n+1} (y_2; c, M^2) \tag{180}$$

$$\varphi_c = \varphi(y_c) = -\frac{1}{w'_c} \lim_{y \to y_c} \left\{ (w - c)^2 \sum_{n=0}^{\infty} \alpha^{2n} k'_{2n+1} (y_c; c, M^2) \right\}$$

$$= -\frac{T_c}{w'_c} \left\{ 1 + \alpha^2 \int_{y_1}^{y_c} \frac{(w-c)^2}{T} dy \int_{y_1}^{y} \left\{ \frac{T}{(w-c)^2} - M^2 \right\} dy + \ldots \right\} \tag{181}$$

It is assumed that the mean velocity temperature profile in any particular case is known for each Mach number. Subject only to the restriction $c < 1 - \frac{1}{M}$, the reflectivity can be calculated for a series of suitable real values of c. These calculations should give some indication of the conditions under which a pure outgoing wave or a pure incoming wave can exist.

(c) Necessary and Sufficient Conditions for the Existence of a Neutral Sonic Disturbance:

If the physical condition that both φ and π/p must be bounded as $y \longrightarrow \infty$ is imposed, then in the case of the neutral sonic disturbance, φ and $\varphi' \longrightarrow 0$ very rapidly as $y \rightarrow \infty$ (sec. 7c) and no energy can be transported into or out of the boundary layer by the disturbance. The necessary condition for the existence of a neutral sonic disturbance is therefore (sec. 8)

$$c = c_s = 1 - \frac{1}{M} \qquad (\alpha \neq 0) \tag{182}$$

Unlike the case of the neutral subsonic disturbance (sec. 9a), the condition $c = c_s$ is not entirely sufficient for the existence of a neutral sonic disturbance. Because the physical significance of this sonic disturbance is not yet clear, it does not seem worth while to discuss this problem in great detail, although some mathematical results have been obtained. A brief sketch of the arguments and results will be given here. From equations (117) to (120), and equation (138), the solution of the differential equation (112) for $\alpha = 0$ which has the correct asymptotic behavior in this case must be

$$\varphi(y) = (w - c) \int_y^\infty \left\{ \frac{T}{(w - c)^2} - M^2 \right\} dy \tag{183}$$

Therefore,

$$\varphi(0) = - c \int_0^\infty \left\{ \frac{T}{(w - c)^2} - M^2 \right\} dy, \qquad \alpha = 0 \tag{184}$$

and hence,

$$\varphi(0) \lesseqgtr 0 \tag{185}$$

according as

$$\int_0^\infty \left\{ \frac{T}{(w-c)^2} - M^2 \right\} dy \gtreqless 0$$

On the other hand, an argument almost identical with that utilized in the case of the subsonic disturbance (sec. 9a) shows that $\varphi(0;\ \alpha) > 0$ for large values of α, if $c = c_s$. Since $\varphi(0,\ \alpha)$ is a bounded continuous function of α, <u>if</u>

$$\int_0^\infty \left\{ \frac{T}{(w-c)^2} - M^2 \right\} dy > 0$$

$\varphi(0,\ \alpha)$ <u>must vanish for some value of</u> $\alpha > 0$; <u>if</u>

$$\int_0^\infty \left\{ \frac{T}{(w-c)^2} - M^2 \right\} dy = 0$$

<u>a non-trivial solution exists for $\alpha = 0$</u>; if

$$\int_0^\infty \left\{ \frac{T}{(w-c)^2} - M^2 \right\} dy < 0$$

it must first be determined whether or not $\varphi(0,\ \alpha) > \varphi(0,\ 0)$ for <u>all</u> α, before any definite conclusion can be drawn. By employing a modification of the oscillation, or comparison theorem (reference 11), it can be shown that $\varphi(0,\ \alpha_2) > \varphi(0,\ \alpha_1)$, if $\alpha_2 > \alpha_1$, and therefore $\varphi(0,\ \alpha)$ is a monotonic increasing function of α. Hence, <u>if</u>

$$\int_0^\infty \left\{ \frac{T}{(w-c)^2} - M^2 \right\} dy < 0$$

no solution of this type exists[1], with $\alpha \neq 0$. (See end of sec. 9a.)

11. Concluding Discussions

The above investigations of energy relations and the necessary and sufficient conditions for the existence of certain types of disturbance, though incomplete, serve to give a general understanding of the stability problem in an inviscid fluid. Before proceeding to include the effect of viscosity, the significance of the results will be discussed somewhat in detail.

The distribution of the density of angular momentum across the boundary layer is unstable if the quantity $\bar{\rho}^* \dfrac{d\bar{u}^*}{dy^*}$ has an extremum for some positive value of $\bar{u}^* > \bar{u}_0^* \left(1 - \dfrac{1}{M}\right)$, where M is the Mach number for the mean flow outside the boundary layer. From the equations of mean motion, it is not difficult to show that the quantity

$$\frac{d}{dy}(\rho w') = \frac{d}{dy}\left(\frac{w'}{T}\right)$$

will always vanish for some value of $w > 0$ if the solid boundary is insulated, or if heat is being transferred to the fluid across the solid boundary. This can be seen as follows:

The dynamical equation for the mean motion

$$\bar{\rho}^*\left(\bar{u}^* \frac{\partial \bar{u}^*}{\partial x^*} + \bar{v}^* \frac{\partial \bar{u}^*}{\partial y^*}\right) = \frac{\partial}{\partial y^*}\left(\bar{\mu}^* \frac{\partial \bar{u}^*}{\partial y^*}\right)$$

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[1]Subsequent investigation has shown that this is the case.

gives

$$\left(\frac{\partial^2 \bar{u}^*}{\partial y^{*2}}\right)_1 = -\frac{m}{\bar{T}^*}\left(\frac{\partial \bar{T}^*}{\partial y^*}\right)_1 \left(\frac{\partial \bar{u}^*}{\partial y^*}\right)_1$$

if

$$\frac{\bar{\mu}^*}{\bar{\mu}_0^*} = \left(\frac{\bar{T}^*}{\bar{T}_0^*}\right)^m \qquad (m = 0.76 \text{ for air})$$

where the subscript "1" refers to the wall. If heat is transferred to the fluid at the wall, $\left(\frac{\partial \bar{T}^*}{\partial y^*}\right)_1 < 0$ and hence $\left(\frac{\partial^2 \bar{u}^*}{\partial y^{*2}}\right)_1 > 0$. Therefore, the quantity

$$\frac{\partial}{\partial y^*}\left(\frac{1}{\bar{T}^*}\frac{\partial \bar{u}^*}{\partial y^*}\right) = \frac{1}{\bar{T}^*}\left(\frac{\partial^2 \bar{u}^*}{\partial y^{*2}} - \frac{\partial \bar{u}^*}{\partial y^*}\frac{1}{\bar{T}^*}\frac{\partial \bar{T}^*}{\partial y^*}\right)$$

must be positive for $y^* = y_1^*$. Thus, $\frac{1}{\bar{T}^*}\frac{\partial \bar{u}^*}{\partial y^*}$ increases from some positive value as y increases from y_1^*. But also it is known that it approaches zero as y^* becomes infinite. Hence, $\frac{1}{\bar{T}^*}\frac{\partial \bar{u}^*}{\partial y^*}$ has a maximum at some point $y^* > y_1^*$; that is, $\frac{d}{dy}\left(\frac{w'}{T}\right)$ vanishes for some $w > 0$.

If the solid boundary is insulated, $\left(\frac{\partial T^*}{\partial y^*}\right)_1 = 0$, and $\left(\frac{\partial^2 \bar{u}^*}{\partial y^{*2}}\right)_1 = 0$. The above argument yields no conclusive result. By differentiating the dynamical equation once more with respect to y^*, and utilizing the equation of continuity for mean motion,

$$\frac{\partial \bar{u}^*}{\partial x^*} + \frac{\partial \bar{v}^*}{\partial y^*} = -\frac{1}{\bar{\rho}^*}\left(\bar{u}^*\frac{\partial \bar{\rho}^*}{\partial x^*} + \bar{v}^*\frac{\partial \bar{\rho}^*}{\partial y^*}\right)$$

it is found that

$$\left(\frac{\partial^3 \bar{u}^*}{\partial y^{*3}}\right)_1 = \frac{\sigma m}{C_p \bar{T}_1^*}\left(\frac{\partial \bar{u}^*}{\partial y^*}\right)^3 > 0$$

Thus, $\frac{\partial^2 \bar{u}^*}{\partial y^{*2}}$ must be positive for some $y^* > y_1^*$; and since $\frac{\partial \bar{T}^*}{\partial y^*} < 0$, for $y^* < y_1^*$, it follows that the quantity $\frac{\partial}{\partial y^*}\left(\frac{1}{\bar{T}^*}\frac{\partial \bar{u}^*}{\partial y^*}\right)$ must be positive for some $y^* > y_1^*$. Hence, the essential conditions for the last case also hold in this case. The same conclusion is therefore obtained.

However, if heat is withdrawn from the fluid at the solid boundary $\left(\frac{\partial \bar{T}^*}{\partial y^*}\right)_1 > 0$ and $\left(\frac{\partial^2 \bar{u}^*}{\partial y^{*2}}\right)_1 < 0$. The signs of the quantities $\frac{\partial \bar{T}^*}{\partial y^*}$ and $\frac{\partial^2 \bar{u}^*}{\partial y^{*2}}$ will remain unchanged as y^* increases from y^*_1 in general.[1] Hence, the quantity $\frac{d}{dy}\left(\frac{w'}{T}\right)$ remains negative and will not vanish.

Therefore, for $M < 1$, if the boundary is insulated or if heat is brought into the fluid, it is certain that the laminar boundary layer in a compressible fluid will be relatively less stable than the isothermal Blasius boundary layer in an incompressible fluid, as far as the inertial forces are concerned. If heat is taken from the fluid, the flow will be more stable. Although these conclusions can probably be extended to the

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[1] Except when the quantity $\left[\left(\bar{T}_s^* - \bar{T}_1^*\right)\sigma^{1/2} + \left(\bar{T}_o^* - \bar{T}_1^*\right)\right] \Big/ \left(\bar{T}_s^* - \bar{T}_o^*\right)$ is very small, where $\bar{T}_s^*$ = stagnation temperature.

case when the velocity of the mean flow outside the boundary layer is only slightly supersonic, no statement can as yet be made for the general supersonic case.[1]

However, the critical Reynolds number defined in terms of free-stream quantities may not necessarily be decreased by heating the solid boundary. For in the viscous solutions of section 3, it is the kinematic coefficient of viscosity near the solid boundary that enters. This coefficient is increased by heating, thus leading to greater stability. Whether the minimum critical Reynolds number for any compressible-fluid boundary layer at any Mach number will be greater or less than the value for the Blasius profile can be determined only by actual calculation. This question will be settled for several representative cases in a forthcoming report by some numerical work following methods to be discussed in the next part of this report.

In a recent report (reference 6), Allen and Nitzberg suggested that the "proper" Reynolds number should be based upon the kinematic viscosity at the solid boundary. For small values of c^*, this is not very different from that at the critical layer. However, they have assumed that the critical Reynolds number $\left(\frac{\bar{u}^* \delta^*}{\bar{\nu}_{11}{}^*}\right)_{cr}$ is equal to the critical Reynolds number for the Blasius profile. For the case of insulated solid boundaries (e. g., airfoil surfaces), their value of $\left(\bar{u}^* \delta^* / \bar{\nu}_{10}^*\right)_{cr}$ may therefore be too high.

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[1]For example, for $M > 1.5$, it may not be possible for a subsonic characteristic-oscillation to exist in certain cases, because in addition to satisfying the equations of motion and the boundary conditions, it must also satisfy the condition $c^* > 1 - 1/M$.

III - STABILITY IN A VISCOUS CONDUCTIVE GAS

12. General Considerations and Methods of Numerical Calculations for the Stability in a Viscous Fluid

The foregoing inviscid investigations serve to illustrate the general behavior of the pressure and inertial forces in the control of the stability of the flow of a compressible fluid. These results can therefore be used as a guide in the investigation of the stability in a real fluid at large Reynolds numbers. In the case of the incompressible fluid, very valuable information has been obtained by consideration of a modification of the results in an inviscid fluid by the effect of viscosity. The general conclusion has been reached that the effect of viscosity is essentially destabilizing at very large Reynolds numbers; and it has been possible to obtain the asymptotic behavior of the neutral stability curve for large values of the Reynolds number, also to give a quick approximate estimation of the minimum critical Reynolds number and indeed to compute the complete curve of neutral stability. In the present case, corresponding developments should also be possible, but the results evidently depend upon the Mach number. Any computation of the curve of neutral stability must be carried out for each value of the Mach number of the free stream.

Owing to the limitations of time, it has not been possible to carry out these computations, The authors, however, laid down the general plan of the calculation of the neutral curve of stability, and repeated the calculation of Tietjens function. Some of the numerical values turn out to be slightly different from those originally given by Tietjens (table I, fig. 6). They agree very closely with the results of Schlichting's later calculations (table 2, p. 73, reference 5).

A method of numerical calculation very similar to that used in the incompressible case will be outlined below. It enables the curve of neutral stability to be computed for each Mach number as soon as the distributions of velocity and temperature are known for that Mach number. Several such distributions have been obtained by Kármán and Tsien (reference 12), by Crocco (reference 13), by Emmons and Brainerd (reference 14), and by Hantzsche and Wendt (reference 15).

Method of numerical calculation.- The calculation of the neutral curve depends upon a proper evaluation of the function $E(\alpha,c,M^2)$ occurring in (105). According to (108), its evaluation depends upon the evaluation of φ_{ij} and φ_{ij}' $(i,j = 1,2)$. To evaluate these functions, the inviscid solutions (117) and (118) are used. After a little calculation, there is obtained

$$\left\{\begin{aligned} \varphi_{11} &= -c \\ \varphi'_{11} &= w'_1 \end{aligned}\right. \qquad \left\{\begin{aligned} \varphi_{21} &= 0 \\ \varphi'_{21} &= -c\left\{\frac{T_1}{c^2} - M^2\right\} \end{aligned}\right. \tag{186}$$

where the subscript $(\ )_1$ denotes values at the wall $y = y_1$. The values of y_2 depend upon the integrals h and k of (119) and (120). Then

$$\left.\begin{aligned} \varphi_{12} &= (1 - c) \sum_{n=o}^{\infty} \alpha^{2n} H_{2n} (c,M^2) \\ \varphi_{22} &= (1 - c) \sum_{n=o}^{\infty} \alpha^{2n} K_{2n+1} (c,M^2) \\ \varphi'_{12} &= (1-c)\left\{\frac{1}{(1-c)^2} - M^2\right\} \sum_{n=o}^{\infty} \alpha^{2n} H_{2n-1} (c,M^2) \\ \varphi'_{22} &= (1-c)\left\{\frac{1}{(1-c)^2} - M^2\right\} \sum_{n=o}^{\infty} \alpha^{2n} K_{2n} (c,M^2) \end{aligned}\right\} \tag{187}$$

where

$$\left.\begin{aligned} H_{2n} (c,M^2) &= h_{2n} (y_2,c,M^2) \\ K_{2n+1}(c,M^2) &= k_{2n+1} (y_2,c,M^2) \\ H_{2n-1} (c,M^2) &= \left\{\frac{1}{(1-c)^2} - 1\right\}^{-1} h'_{2n} (y_2,c,M^2) \\ K_{2n} (c,M^2) &= \left\{\frac{1}{(1-c)^2} - 1\right\}^{-1} k'_{2n+1} (y_2,c,M^2) \end{aligned}\right\} \tag{188}$$

Substituting (187) into (108) gives

$$E(\alpha,c,M^2) = \frac{w_1' \, (\varphi_{22}' + \beta\,\varphi_{22})}{w_1' \, (\varphi_{22}' + \beta\varphi_{22}) + \left\{\frac{T_1}{c}\right\}\left\{\varphi_{12}' + \beta\varphi_{12}\right\}} \; \frac{1}{1+\lambda} \tag{189}$$

where $\lambda = \lambda(c)$ is defined by

$$w_1' \, (y_1 - y_c) = -c(1 + \lambda) \tag{190}$$

By introducing the function

$$\underline{F}(z) = \frac{1}{1 - F(z)} \tag{191}$$

and using the relation (189), (105) may be reduced to the form

$$\underline{F}(z) = \frac{(1+\lambda)\ \ (u + iv)}{1 + \lambda\ \ \ (u + iv)} \quad \text{with} \quad u + iv = 1 + \frac{w_1' \, c(\varphi_{22}' + \beta\varphi_{22})}{(T_1)(\varphi_{12}' + \beta\varphi_{12})} \tag{192}$$

It is noted that the quantities φ_{22}' and φ_{22} involve the integrals K_{2n} and K_{2n+1}. These integrals involve

$$\int_{y_1}^{y} \left\{\frac{T}{(w - c)^2} - M^2\right\} dy = K_1 - \int_{y}^{y_2} \left\{\frac{T}{(w - c)^2} - M^2\right\} dy$$

in the first step of integration. By substituting the right-hand-side expression into $\varphi_{22}' + \beta\varphi_{22}$, it is not difficult to verify that the terms involving K_1 combine to give $\varphi_{12}' + \beta\varphi_{12}$. Hence, it is convenient to write

$$\varphi_{22}' + \beta\varphi_{22} = (\varphi_{12}' + \beta\varphi_{12})\, K_1 - \Phi$$

Substituting this into (192) gives finally (with $y_2 - y_1 = 1$)

$$u + iv = 1 + \frac{w_1' \, c}{T_1}\left\{\int_{y_1}^{y_2} \frac{T}{(w - c)^2}\, dy - M^2\right\} + \frac{w_1' \, c\, \Phi}{(T_1)(\varphi_{12}' + \beta\varphi_{12})} \tag{193}$$

The function $\displaystyle\int_{y_1}^{y_2} \frac{T}{(w - c)^2}\, dy$ and the integrals involved in φ_{12}, φ_{12}', and Φ may be evaluated by methods similar to those used in the incompressible case. The significance of bringing the final equation in the form (193) is that the imaginary part of the right-hand side

is mainly contribuced by the term involving $\int_{y_1}^{y_2} T(w - c)^{-2}\, dy$ the

imaginary part of which is $-\pi \frac{T_c}{w_c'^2}\left\{\frac{w_c''}{w_c'} - \frac{T_c'}{T_c}\right\}$. This can be easily calculated. Thus, using the fact that λ is usually very small gives, approximately,

$$\underline{F}_i(z) = -\pi \frac{T_c}{w_c'^2}\left(\frac{w_c''}{w_c'} - \frac{T_c'}{T_c}\right)\frac{w_1' c}{T_1} \tag{194}$$

where $\underline{F}_i(z)$ is the imaginary part of $\underline{F}(z)$. The relation (194) would give a correspondence between c and z. From this, the value of αR can be easily calculated by means of (107). For more accurate calculations, use is made of the relations

$$\left.\begin{aligned}
\underline{F}_r(z) &= (1 + \lambda)\left\{u(1 + \lambda u) - \lambda v^2\right\}\left\{(1 + \lambda u)^2 + (\lambda v)^2\right\}^{-1} \\
\underline{F}_i(z) &= (1 + \lambda)v\left\{(1 + \lambda u)^2 + (\lambda v)^2\right\}^{-1}
\end{aligned}\right\} \tag{195}$$

where $\underline{F}_r(z)$ is the real part of $\underline{F}(z)$. Using (194) as the initial approximation, a method of successive approximations can be developed exactly as in the incompressible case for the calculation of α, R for given values of c. The complete numerical calculation will be carried out for a few typical cases.

California Institute of Technology,
Pasadena, Calif., November 15, 1944.

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Table I.- Functions F(z) and $\underline{F}(z)$

z	F_r	F_i	$\underline{F}_r$	$\underline{F}_i$
1.0	0.89161	-0.35025	0.80630	-2.60557
1.2	.78969	-.27310	1.77012	-2.29854
1.4	.71970	-.21213	2.26836	-1.71669
1.6	.66931	-.16009	2.44985	-1.18600
1.8	.63143	-.11274	2.48104	-.75892
2.0	.60144	-.06741	2.43927	-.41253
2.2	.57599	-.02226	2.35196	-.12348
2.4	.55230	-.02395	2.22724	+.11916
2.6	.52773	-.07203	2.06929	.31558
2.8	.49952	+.12220	1.88566	.46043
3.0	.46456	.17391	1.68938	.54872
3.2	.41947	.22520	1.49726	.58082
3.4	.36110	.27193	1.32516	.56401
3.6	.28802	.30705	1.18429	.51074
3.8	.20352	.32130	1.07982	.43560
4.0	.11800	.30721	1.01118	.35220
4.2	.04698	.26559	.97361	.27133
4.4	.00240	.20811	.96056	.20038
4.6	.02160	.14475	.95989	.13601
4.8	.01477	.09875	.97659	.09503

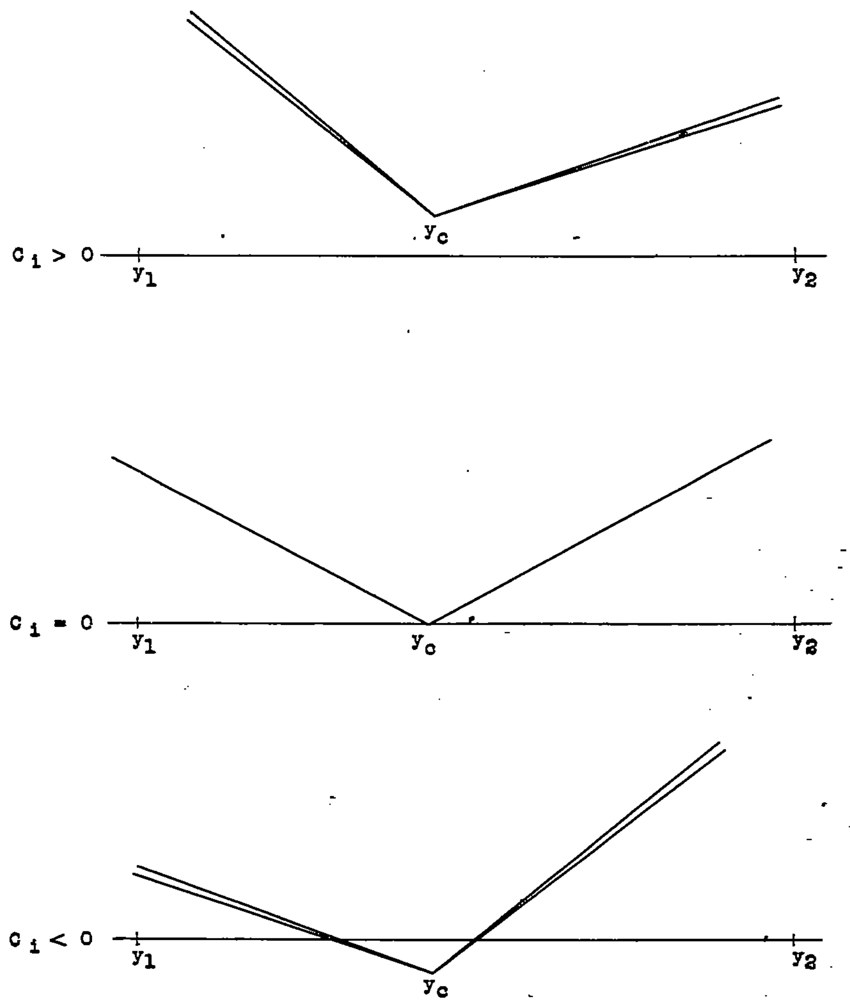

Figure 1.- Region of validity of the asymptotic expansions of the regular solutions for $c_i \gtreqless 0$.

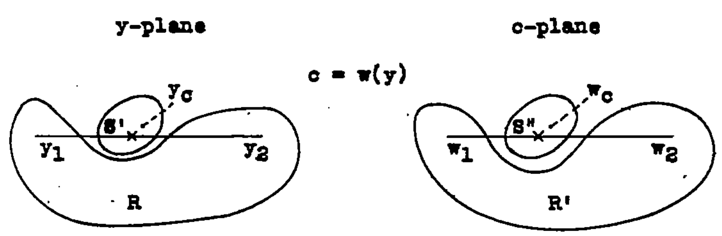

Figure 2.- Region of analyticity of the inviscid solutions.

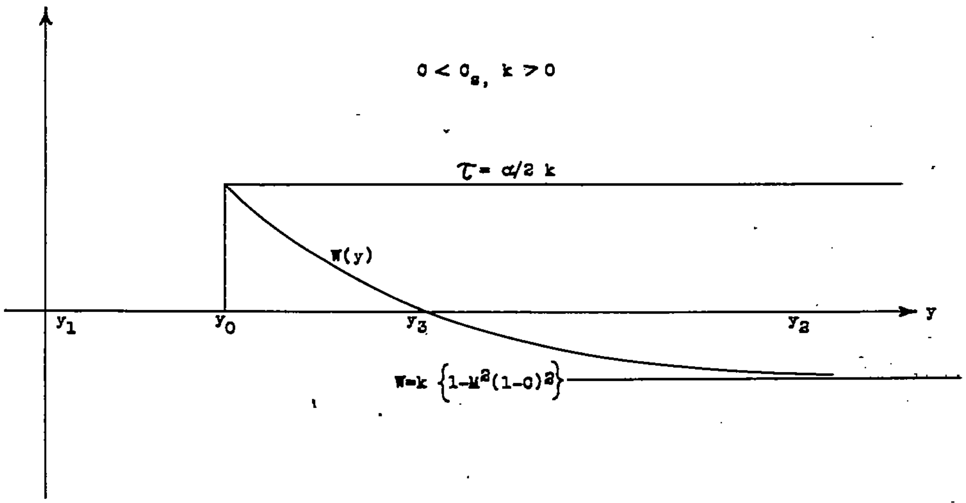

Figure 5.- Distribution of Wronskian, $W(y)$, and shear stress, $\tau(y)$, for the inviscid, neutral supersonic disturbance.

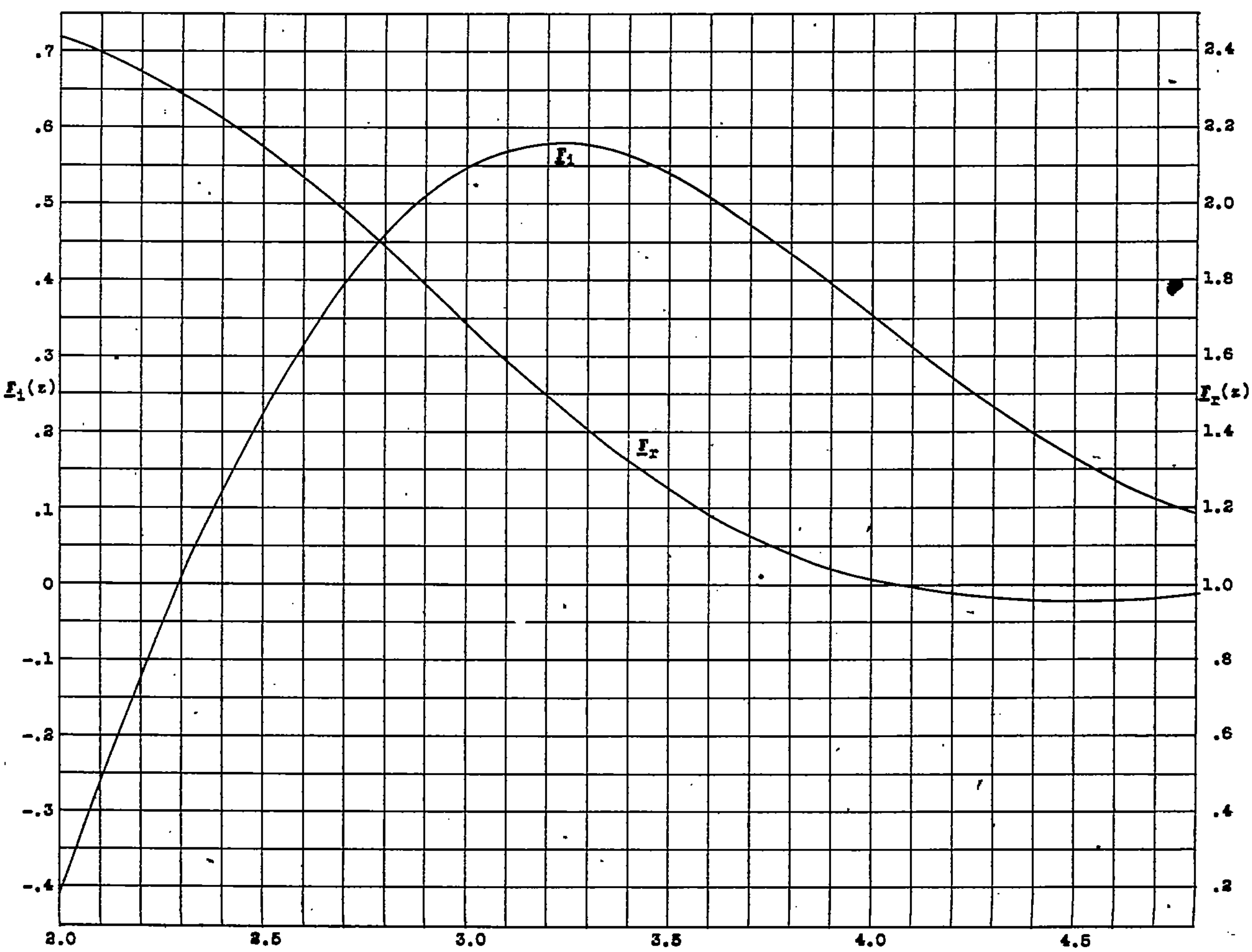

Figure 6.- The functions $F_r(z)$ and $F_i(z)$.